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ORIGINAL ARTICLE

Adaptive control of nonlinear fractional-order systems using T–S fuzzy method

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Abstract Owing to the superior capability of fractional differential equations in modeling and characterizing accurate dynamical properties of many high technology real world systems, the design and control of fractional-order systems have captured lots of attention in recent decades. In this paper, an adaptive intelligent fuzzy approach to controlling and stabilization of nonlinear non-autonomous fractional-order systems is proposed. Since dynamic equations of applied fractional-order systems usually contain various parameters and nonlinear terms, the Takagi–Sugeno (T–S) fuzzy models with if-then rules are adopted to describe the system dynamics. Also, as the nonlinear system parameters are assumed to be unknown, adaptive laws are derived to estimate such fluctuations. Simple adaptive linear-like control rules are developed based on the T–S fuzzy control theory. The stability of the resulting closed loop system is guaranteed by Lyapunov's stability theory. Two illustrative numerical examples are presented to emphasize the correct performance and applicability of the proposed adaptive fuzzy control methodology. It is worth to notice that the proposed controller works well for stabilization of a wide class of either autonomous nonlinear uncertain fractional-order systems or non-autonomous complex systems with unknown parameters.

Keywords T–S fuzzy control · Fractional-order systems · Adaptive approach · Lyapunov's theory · Intelligent control

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1 Introduction

Fractional-Order (FO) calculus has a long history and a retrospect as long as three centuries, and it has attracted increasing attentions in physics and engineering in recent years [1–4]. Nowadays, it has been known that the nature of many real phenomena can be perfectly characterized and modeled using fractional differential equations and many dynamical systems in various applied fields, such as biology [5], medicine [6], physics [7, 8], electro-mechanics [9] and social sciences [10, 11].

In the control theory field, it is important to force the system trajectory into a desired command. In this content, regarding to the non-integer-order systems, various control methods have been utilized to achieve good performances for fractional-order nonlinear systems(FONSs) [12–18]. Besides, researches keep making great efforts in expanding various control design methods, such as sliding mode control [19, 20], pinning control [21], adaptive control [22–24], intelligent control [25], and predictive control [26], to realize effective controllers for FONSs. For instance, in [27, 28], the author has introduced sliding mode control schemes to control and stabilize uncertain fractional-order systems via Lyapunov's stability theory. In [29], the author attempts to overcome the discontinuity of the standard Gaussian noise, Levy noise and Markov switching by using the Razumikhin method and Lyapunov functions. In [30, 31], He et al. have been utilized an adaptive impedance control for an n-link robotic manipulator with input saturation. In [32], the authors have introduced adaptive fuzzy neural network controls for a constrained robot with unknown system dynamics. In [33], the internal forces have been applied to guarantee effective cooperation between the dual arms. In [34], an optimal control method has been utilized to control FONSs. The control and stabilization of the fractional-order Liu system

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has been performed via feedback control scheme in [35]. In [36], Zhu et al. have studied the issue of input-to-state stability for a class of impulsive stochastic Cohen–Grossberg neural networks with mixed delays. In [37], Zheng has designed a prediction-based feedback controller to stabilize a fractional-order nonlinear system around its original equilibrium point. In [38], the authors have addressed the asymptotic stability of Markov switched stochastic differential equations. In [39], a robust fuzzy control method has been investigated for fractional-order hydro-turbine governing system in the presence of random disturbances.

As one of the applied techniques for modeling of dynamical systems, Takagi and Sugeno have proposed a fuzzy logic strategy which is known as T–S fuzzy method [40]. The T–S fuzzy technique has been frequently used for mathematical simplicity of analysis by approximating complex nonlinear systems. This method can be expressed as a nonlinear dynamical system with a small number of IF-THEN rules. The overall model of the system is then achieved by fuzzy blending of these linear models [41]. Recent studies have addressed the development on T–S fuzzy control approach as a widely used control technology for integer-order systems [42–46]. Consequently, the study of FONSs has been extended to T–S fuzzy models [47]. To this end, some researches have investigated the applicability of intelligent fuzzy control methodology to fractional-order systems. In [48], the authors have investigated the finite-time stabilization for a class of high-order stochastic nonlinear systems in strict-feedback forms. In [49], the author has addressed the problem of fractional-order output feedback controller design for fractional T-S fuzzy systems with deterministic parameters and immeasurables premise variables. In [50], a fractional-order T–S fuzzy control method has been presented for a class of non-integer-order nonlinear systems. Lin et al. [51] have investigated the stabilization problem of fractional-order T–S fuzzy systems via static output feedback control. In [52], the problem of the moment exponential stability for a class of impulsive stochastic functional differential equations with Markovian switching has been presented. In [53], Wang et al. have focused on the stabilization of fractional-order interval systems with uncertain parameters via the generalized T–S fuzzy models. In [54], the problem of designing C^1 or C^∞ controllers for a class of stochastic nonlinear systems has been investigated by using the back stepping method. He et al. [55] have proposed a neural network controller to suppress the vibration of a flexible robotic manipulator system with input dead-zone. In [56], the stabilization problem for uncertain fractional-order unstable systems has been studied for the case where the fractional order satisfies $0 < \beta < 1$ and $1 \leq \beta < 2$.

Most of the above-mentioned works have been designed for T–S fuzzy fractional-order systems with fully known parameters. However, in practice, there are usually some unknown constant parameters in the dynamics of the fractional-order systems. Tackling such uncertainties is important in both theory and application. In addition, the previous papers have restricted their focus on autonomous systems with no time-varying external disturbances. While, most of the practical systems are usually non-autonomous and external disturbances are unavoidably present in the dynamical system. In other words, such disturbances and/or measurement noises can make a system non-autonomous and uncertain which result in a complex system. It is known that dealing with time-varying NFOSs is more difficult compared to the autonomous cases. However, to our best knowledge, there is a little works in the literature with a focus on controlling non-autonomous uncertain fractional-order systems in spite of both external perturbations and unknown parameters.

Motivated by the above discussions, this paper concerns the problem of robust stabilization of non-autonomous nonlinear fractional-order systems in the presence of unknown constant system matrices. We propose an intelligent fuzzy model-based adaptive approach for nonlinear fractional-order systems with unknown parameters. In this regard, a fractional T–S fuzzy model is developed for a wide class of FONSs. Then, based on the Lyapunov theory and adaptive control concept, simple linear-type adaptive fuzzy rules are proposed to assure the asymptotic stability for the origin equilibrium point of the fluctuated system. To estimate the system's unknown parameters, some adaptive laws are derived and used as a part of the controller. Based on mathematical proofs, numerical simulations show the superior performance of the proposed T–S fuzzy control scheme for both autonomous and non-autonomous uncertain fractional systems. The main contribution of this paper is dealing with uncertain non-autonomous nonlinear fractional-order systems with external disturbances via a robust semi-linear adaptive fuzzy approach. In fact, in our design, one can use the well-known modern linear control methods to derive a suitable controller with improving the stability of the system. Another contribution of this paper is to separate the unknown time-dependent and time-independent terms of the system in order to estimate them by some simple adaptive rules. This can improve the analysis and design approach of the control technique and can result in a low complexity control scheme.

The rest of this article is organized as follows. In Sect. 2, some necessary definitions of fractional-order calculus and preliminaries for T–S fuzzy modeling of nonlinear systems are given. The proposed fuzzy model-based adaptive approach is designed in Sect. 3. The validity and the effectiveness of the proposed control method are examined through numerical examples in Sect. 4. Finally, Sect. 5 presents the conclusions of this study.

2 Preliminaries and system description

This section, presents some basic definitions about fractional-order differential equation. Then, it addresses two necessary lemmas for FO systems. Some concepts and techniques related to the stability of FONSs are also discussed. Finally, definition of the fractional-order T–S fuzzy model (FOTSFM) is presented.

2.1 Fractional calculus

Definition 2.1 [3] The αth fractional-order Riemann Liouville integration of function $f(t)$ is defined by

$$ {}_{t_0}I_t^{\alpha} f(t) = {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{1} $$

where $\Gamma(\cdot)$ is the Gamma function which is defined as

$$ \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \tag{2} $$

Definition 2.2 [3] The Caputo fractional derivative of order $\alpha \in R^{+}$of a function $f(t)$ is defined as follows.

$$ {}_a^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{3} $$

where $n = \min\{k \in N, k > \alpha\}$.

Definition 2.3 [3] The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $f(t)$ is defined as the nth derivative of fractional integral (1) of order $n-\alpha$which is given as below.

$$ {}_{t_0}^{RL}D_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha}.f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(t)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{4} $$

where n is the smallest integer larger than or equal to α.

Using the Caputo derivative, an uncertain fractional-order system with control inputs can be defined by

$$ {}_a^cD_t^{\alpha} x(t) = f(x, t, \Delta, u) \tag{5} $$

where $\alpha \in (0, 1)$, x stands for the states of the system, t shows the time, Δ represents uncertain variables of the system and u is the control input.

Lemma 2.1 [57] *Let $x(t) \in R^n$ be a vector of differentiable function. Then, for any time instant $t \geq t_0$ the following inequality holds.*

$$ \frac{1}{2} {}_a^cD_t^{\alpha} \left[x^T(t) \cdot P \cdot x(t)\right] \leq x^T(t) \cdot P \cdot {}_a^cD_t^{\alpha} x(t), \forall \alpha \in (0, 1) \tag{6} $$

where $P \in R^{n\times n}$ is a constant, square, symmetric and positive definite matrix.

Lemma 2.2 [57] *Let $A(t) \in R^{n\times n}$ be a time-varying differentiable matrix. Then, for any time instant $t \geq t_0$ the following inequality holds.*

$$ {}_a^cD_t^{\alpha} \left[tr\left\{A^T(t) \cdot P \cdot A(t)\right\}\right] \leq 2tr\left\{A^T(t) \cdot P \cdot {}_a^cD_t^{\alpha} A(t)\right\}, \forall \alpha \in (0, 1) \tag{7} $$

Definition 2.4 [58] A continuous function $\Upsilon : [0, t) \to [0, \infty)$ is said to belong to class-k if it is strictly increasing and $\Upsilon(0) = 0$.

Theorem 2.1 [58] *Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system (5). Assume that there exists a Lyapunov function $V(x(t), t)$ and class-K functions $\Upsilon_i\,(i = 1, 2, 3)$ satisfying*

$$ \begin{cases} \Upsilon_1(\|x\|) \leq V(x(t), t) \leq \Upsilon_2(\|x\|) \\ {}_a^cD_t^{\beta} V(x(t), t) \leq -\Upsilon_3(\|x\|) \end{cases} \tag{8} $$

where $\beta \in (0, 1)$, then the origin of the system (5) is asymptotically stable.

2.2 Generalized T–S fuzzy model of fractional-order systems

The generalized T–S fuzzy model for fractional-order systems is obtained by extending the conventional T–S fuzzy model [40]. In the generalized T–S fuzzy model, the system dynamics are represented by local fractional-order linear models. Then, the overall model of the system is achieved by fuzzy blending of the linear models. Suppose that the generalized T–S fuzzy model is given in the following form.

Rule i If $z_1(t)$ is M_{i1} and $\ldots z_p(t)$ is M_{ip}.

Then $D^{\alpha}x(t) = A_i x(t)\ i = 1, \ldots, r$. where $M_{ij}(j = 1, \ldots, p)$ is the fuzzy set, r denotes the number of IF-THEN rules, $x(t) \in R^n$ is the state vector $A_i \in R^{n\times n}$, $z_1(t) \cdots z_p(t)$ are the premise variables and $\alpha(0 < \alpha < 1)$represents the fractional order.

The final output of the generalized T-S fuzzy model is inferred as follows.

$$ D^{\alpha}x(t) = \frac{\sum_{i=1}^{r} w_i(z(t)) \cdot A_i x(t)}{\sum_{i=1}^{r} w_i z(t)} \tag{9} $$

where $z(t) = \left(z_1(t), ..., z_p(t)\right)$ and $w_i(z(t)) = \prod_{j=1}^{p} M_{ij}\left(z_j(t)\right)$, with $M_{ij}(z(t))$ being the grade of membership of $z_j(t)$ in M_{ij} satisfying the following conditions.

$$\begin{cases} \sum_{i=1}^{r} w_i(z(t)) > 0 \\ w_i(z(t)) \geq 0 \end{cases} \tag{10}$$

By introducing $h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}$ instead of $w_i(z(t))$, the above expression is rewritten as follows.

$$D^{\alpha} x(t) = \sum_{i=1}^{r} h_i(z(t)) \cdot A_i x(t) \tag{11}$$

Note that the following conditions should be held.

$$\begin{cases} \sum_{i=1}^{r} h_i(z(t)) = 1 \\ h_i(z(t)) \geq 0 \end{cases} \tag{12}$$

where $h_i(z(t))$ can be regarded as the normalized weights of the IF-THEN rules.

3 Main results

In this section, the control problem of T–S fuzzy uncertain fractional-order is formulated. Then, an adaptive control scheme is proposed to stabilize the FOTSFMs with unknown parameters.

Suppose that the nonlinear non-autonomous system (5) with uncertain terms and control inputs can be modeled by a FOTSFM as follows.

Rule i If $z_1(t)$ is M_{i1},…, and $z_p(t)$ is M_{ip}, THEN

$$D^{\alpha} x(t) = A_i x(t) + A_{ui} x(t) + B_{ui} \zeta(t) + B_i \zeta(t) + u(t) \tag{13}$$

where $x(t) \in R^n$ is the state of the nonlinear system, $z(t) = \left[z_1(t), \ldots, z_p(t)\right]^T$ denotes a vector of the premise variables, $M_{ij}(j = 1, \ldots, p)$ and $(i = 1, \ldots r,)$ are fuzzy sets, r represents the number of fuzzy rules, $A_i$and B_i are some constant matrices of compatible dimensions, A_{ui} and B_{ui} are unknown parameters of the system, $\zeta(t) \in R^m$ may denote the oscillated force or a disturbance term in the system dynamics, and $u(t) \in R^n$ is the control input vector.

The final output of the fractional-order T-S fuzzy system is inferred as follows.

$$D^{\alpha} x(t) = \sum_{i=1}^{r} h_i(z(t)) \left\{A_i x(t) + A_{ui} x(t) + B_i \zeta(t) + B_{ui} \zeta(t)\right\} + u(t) \tag{14}$$

To realize the global asymptotical stabilization of the origin equilibrium point for the system (14), the following adaptive fuzzy controller is proposed in this paper.

Control Rule i If $z_1(t)$ is M_{i1},…, $z_p(t)$ is M_{ip}, THEN

$$u_i(t) = -\left(\hat{A}_{ui} x(t) + \hat{B}_{ui} \zeta(t) + K_i x(t) + L_i \zeta(t)\right) \tag{15}$$

where K_i and L_i are the feedback gain matrices with appropriate dimension to be designed later, $\hat{A}_{ui}$ and $\hat{B}_{ui}$ are estimations of the unknown parameters A_{ui} and B_{ui}, respectively which are generated by some adaptive laws given below.

Based on Eqs. (14) and (15) the inferred control law is represented by

$$u(t) = -\sum_{i=1}^{r} h_i(z(t)) \left(\hat{A}_{ui} x(t) + \hat{B}_{ui} \zeta(t) + K_i x(t) + L_i \zeta(t)\right) \tag{16}$$

And, the adaptive laws are designed by

$$D^{\alpha} \hat{A}_{ui} = \sigma_{A_{ui}} \cdot h_i(z(t)) \cdot x(t) \cdot x^T(t) \tag{17}$$

$$D^{\alpha} \hat{B}_{ui} = \sigma_{B_{ui}} \cdot h_i(z(t)) \cdot x(t).\zeta^T(t) \tag{18}$$

where $\sigma_{A_{ui}}$ and $\sigma_{B_{ui}}$ are positive constant adaption gains.

Therefore, the final closed-loop system containing the control law (16) is obtained as follows.

$$\begin{aligned} D^{\alpha} x(t) = \sum_{i=1}^{r} h_i(z(t)) \{ & \left(A_i - K_i\right) x(t) + \left(A_{ui} - \hat{A}_{ui}\right) x(t) \\ & + \left(B_{ui} - \hat{B}_{ui}\right) \zeta(t) + \left(B_i - L_i\right) \zeta(t) \} \end{aligned} \tag{19}$$

Defining $\tilde{A}_{ui} = A_{ui} - \hat{A}_{ui}$ and $\tilde{B}_{ui} = B_{ui} - \hat{B}_{ui}$, the system (19) is changed into the following equality.

$$\begin{aligned} D^{\alpha} x(t) = \sum_{i=1}^{r} h_i(z(t)) \{ & \left(A_i - K_i\right) x(t) + \tilde{A}_{ui} x(t) \\ & + \tilde{B}_{ui} \zeta(t) + \left(B_i - L_i\right) \zeta(t) \} \end{aligned} \tag{20}$$

In order to prove the stability of the origin equilibrium point of the system (20), the following theorem is presented.

Theorem 3.1 *Consider the non-autonomous fractional-order fuzzy system* (14) *with unknown parameters. If this system is controlled by the adaptive controller* (16)–(18), *then the system state trajectories will converge to zero asymptotically*.

Proof Let us select a Lyapunov function candidate in the following form.

$$V(x, A_{ui}, B_{ui}) = x^T(t)\cdot P\cdot x(t) + \sum_{i=1}^{r} tr\left(\frac{\tilde{A}_{ui}^T\cdot P\cdot \tilde{A}_{ui}}{\sigma_{A_{ui}}}\right) + \sum_{i=1}^{r} tr\left(\frac{\tilde{B}_{ui}^T\cdot P\cdot \tilde{B}_{ui}}{\sigma_{B_{ui}}}\right) \tag{21}$$

where P is a positive definite symmetric matrix and $tr(A)$ denotes the trace of a matrix.

Using Lemmas 2.1 and 2.2, the time derivative of V becomes

$$D^\alpha V \le 2x^T(t)\cdot P\cdot D^\alpha x(t) + 2\sum_{i=1}^{r} tr\left(\frac{\tilde{A}_{ui}^T\cdot P\cdot D^\alpha\tilde{A}_{ui}}{\sigma_{A_{ui}}}\right) + 2\sum_{i=1}^{r} tr\left(\frac{\tilde{B}_{ui}^T\cdot P\cdot D^\alpha\tilde{B}_{ui}}{\sigma_{B_{ui}}}\right) \tag{22}$$

With using Eq. (20) and $\tilde{A}_{ui} = A_{ui} - \hat{A}_{ui}$, $\tilde{B}_{ui} = B_{ui} - \hat{B}_{ui}$, we have

$$D^\alpha V \le 2x^T(t)\cdot P\cdot \sum_{i=1}^{r} h_i(z(t))\{(A_i - K_i)x(t) + \tilde{A}_{ui}x(t) + \tilde{B}_{ui}\zeta(t) + (B_i - L_i)\zeta(t)\} + 2\sum_{i=1}^{r} tr\left(\frac{\tilde{A}_{ui}^T P\cdot D^\alpha(A_{ui} - \hat{A}_{ui})}{\sigma_{A_{ui}}}\right) + 2\sum_{i=1}^{r} tr\left(\frac{\tilde{B}_{ui}^T P D^\alpha(B_{ui} - \hat{B}_{ui})}{\sigma_{B_{ui}}}\right) \tag{23}$$

According to the equation $2x^T(t).PA\cdot x(t) = x^T(t)(A^TP + PA)x(t)$, one has

$$\begin{aligned} D^\alpha V \le & \sum_{i=1}^{r} h_i(z(t))\left\{x^T(t)\left((A_i - K_i)^T P + P(A_i - K_i)\right)x(t)\right\} \\ & + \sum_{i=1}^{r} h_i(z(t))\left\{\zeta^T(t)\left((B_i - L_i)^T P + P(B_i - L_i)\right)\zeta(t)\right\} \\ & + 2x^T(t)P\sum_{i=1}^{r} h_i(z(t))\tilde{A}_{ui}x(t) + 2x^T(t)P\sum_{i=1}^{r} h_i(z(t))\tilde{B}_{ui}\zeta(t) \\ & - 2\sum_{i=1}^{r} tr\left(\frac{\tilde{A}_{ui}^T pD^\alpha\hat{A}_{ui}}{\sigma_{A_{ui}}}\right) - 2\sum_{i=1}^{r} tr\left(\frac{\tilde{B}_{ui}^T PD^\alpha\hat{B}_{ui}}{\sigma_{B_{ui}}}\right) \end{aligned} \tag{24}$$

Since for the vectors $x \in \mathbb{R}^{n\times 1}$ and $y \in \mathbb{R}^{n\times 1}$ the equalities $x^Tx = trace(xx^T)$ and $x^Ty = trace(xy^T)$ are always held, one can rewrite the inequality (24) as follows:

$$\begin{aligned} D^\alpha V \le & \sum_{i=1}^{r} h_i(z(t))\left\{x^T(t)\left((A_i - K_i)^T P + P(A_i - K_i)\right)x(t)\right\} \\ & + \sum_{i=1}^{r} h_i(z(t))\left\{\zeta^T(t)\left((B_i - L_i)^T P + P(B_i - L_i)\right)\zeta(t)\right\} \\ & + 2tr\left\{\sum_{i=1}^{r} h_i(z(t))(\tilde{A}_{ui}^T Px(t)x^T(t)) - \sum_{i=1}^{r}\frac{\tilde{A}_{ui}^T PD^\alpha\hat{A}_{ui}}{\sigma_{A_{ui}}}\right\} \\ & + 2tr\left\{\sum_{i=1}^{r} h_i(z(t))(\tilde{B}_{ui}^T Px(t)\zeta^T(t)) - \sum_{i=1}^{r}\frac{\tilde{B}_{ui}^T PD^\alpha\hat{B}_{ui}}{\sigma_{B_{ui}}}\right\} \end{aligned} \tag{25}$$

With substituting Eqs. (17) and (18) in Eq. (25), we have

$$\begin{aligned} D^\alpha V \le & \sum_{i=1}^{r} h_i(z(t))\left\{x^T(t)\left((A_i - K_i)^T P + P(A_i - K_i)\right)x(t)\right\} \\ & + \sum_{i=1}^{r} h_i(z(t))\left\{\zeta^T(t)\left((B_i - L_i)^T P + P(B_i - L_i)\right)\zeta(t)\right\} \\ & + 2tr\left\{\sum_{i=1}^{r} h_i(z(t))(\tilde{A}_{ui}^T Px(t)x^T(t)) - \sum_{i=1}^{r}\frac{\tilde{A}_{ui}^T P\sigma_{A_{ui}}h_i(z(t))x(t)x^T(t)}{\sigma_{B_{ui}}}\right\} \\ & + 2tr\left\{\sum_{i=1}^{r} h_i(z(t))(\tilde{B}_{ui}^T Px(t)x^T(t)) - \sum_{i=1}^{r}\frac{\tilde{B}_{ui}^T P\sigma_{B_{ui}}h_i(z(t))x(t)\zeta^T(t)}{\sigma_{B_{ui}}}\right\} \end{aligned} \tag{26}$$

Since $\sum_{i=1}^{r} h_i(z(t))(\tilde{A}_{ui}^T Px(t)x^T(t)) = \sum_{i=1}^{r}\frac{\tilde{A}_{ui}^T P\sigma_{A_{ui}}h_i(z(t))x(t)x^T(t)}{\sigma_{A_{ui}}}$ and $\sum_{i=1}^{r} h_i(z(t))(\tilde{B}_{ui}^T Px(t)\zeta^T(t)) = \sum_{i=1}^{r}\frac{\tilde{B}_{ui}^T P\sigma_{B_{ui}}h_i(z(t))x(t)\zeta^T(t)}{\sigma_{B_{ui}}}$ are satisfied, one obtains

$$D^\alpha V \le \sum_{i=1}^{r} h_i(z(t))\left\{\begin{array}{l} x^T(t)\left((A_i - K_i)^T P + P(A_i - K_i)\right)x(t) \\ +\zeta^T(t)\left((B_i - L_i)^T P + P(B_i - L_i)\right)\zeta(t) \end{array}\right\} \tag{27}$$

It is obvious that $D^\alpha V \le 0$ is equivalent to

$$\left\{\begin{array}{l} (A_i - K_i)^T P + P(A_i - K_i) < 0 \\ (B_i - L_i)^T P + P(B_i - L_i) < 0 \end{array}\right. \tag{28}$$

Therefore, if there is a positive symmetric matrix P along with chosen K_i and L_i such that the conditions of Eq. (27) are met, then $D^\alpha V \le 0$ is guaranteed and the state trajectories of the non-autonomous fractional-order nonlinear fuzzy system (14) with unknown parameters will converge to zero asymptotically. Hence, the proof is completed.

Remark 1 According to the control input (16) it is seen that the control input is directly proportional to the values of the parameters K_i and L_i. This fact implies that smaller values of K_i and L_i result in a smaller control effort and vice versa. On the other hand, Hence, based on Eq. (28), the parameters K_i and L_i should be chosen regarding the stability inequality (28) to assure an asymptotical convergence. These issues may be taken into account by the designer based on the implementation requirements.

Remark 2 In accordance to the adaptive rules in Eqs. (17) and (18), the values of the update parameters $\hat{A}_{ui}$ and $\hat{B}_{ui}$ are directly proportional to the values of $\sigma_{A_{ui}}$ and $\sigma_{B_{ui}}$. And, the estimation parameters $\hat{A}_{ui}$ and $\hat{B}_{ui}$ affect the magnitude of the control input, straightforwardly. So, larger $\sigma_{A_{ui}}$ and $\sigma_{B_{ui}}$ makes a controller with an expensive control effort. On the other hand, based on Eqs. (17) and (18) the estimation convergence time is controlled via the parameters $\sigma_{A_{ui}}$ and $\sigma_{B_{ui}}$. Therefore, to obtain a fast estimation, one can choose the values of the parameters $\sigma_{A_{ui}}$ and $\sigma_{B_{ui}}$ to be large with considering the effects of the choice on the control effort to not be too large.

Remark 3 It is worth to note that compared to the other existing works in the literature such as [16, 24, 37, 39], the main novelty of the proposed T–S fuzzy fractional controller is that the introduced control method has a simple linear-like structure and it can be easily implemented for both autonomous and non-autonomous systems even when the constant parameters of the system are fully unknown in advance.

4 Simulation results

In this section, in order to show the validity and feasibility of the proposed adaptive control scheme for T–S fuzzy nonlinear systems with unknown parameters, we consider two well-known nonlinear chaotic systems: the fractional-order autonomous Genesio–Tesi [59] and the non-autonomous fractional-order horizontal platform [19]. Both these systems have positive Lyapunov exponent for some values of the fractional order α. However, in the chaos literature, there are also some conservative chaotic systems in which they have a conserved volume of the flow. A chaotic system with positive Lyapunov exponent is conservative if the sum of the system's Lyapunov exponents is zero. The fractional-order systems Genesio–Tesi and horizontal platform may show volume-conserving property for some values of α [19, 59]. The proposed fuzzy approach is still applicable for conservative chaotic fractional-order system.

4.1 Control of autonomous fractional-order Genesio–Tesi system

Consider the Genesio–Tesi nonlinear system whose dynamics is given by [59]

$$\begin{cases} D^\alpha x_1(t) = x_2(t) \\ D^\alpha x_2(t) = x_3(t) \\ D^\alpha x_3(t) = -b_1x_1(t) - b_2x_2(t) - b_3x_3(t) + b_4x_1^2(t) \end{cases} \tag{29}$$

where $b_1 = 6, b_2 = 2.92, b_3 = 1, b_4 = 1$ are the parameters of the Genesio–Tesi system. Since this system may display stable behaviors for the values less than 0.95, here, α is set 0.98 to ensure an unstable behavior for the system. Now, for applying the designed adaptive control approach, the T–S fuzzy model representation of the Genesio–Tesi (29) is needed. By defining two fuzzy sets and assumption of $x_1 \in (-d, d)$ and $d = 20$, one can obtain the following fuzzy system for the Genesio–Tesi (28).

Rule 1 If $x_1(t)$ is $M_1(x(t))$ Then $D^\alpha x(t) = A_1x(t) + u(t)$.

Rule 2 If $x_2(t)$ is $M_2(x(t))$ Then $D^\alpha x(t) = A_2x(t) + u(t)$

$$D^\alpha x = \sum_{i=1}^{2} h_i(z(t))\left\{A_ix(t) + A_{ui}x(t)\right\} + u(t) \tag{30}$$

where $x(t) = \left[x_1(t), x_2(t), x_3(t)\right]^T$, $u(t) = \left[u_1(t), u_2(t), u_3(t)\right]^T$ and A_{ui}, $A_i$are the unknown parameters of the system defined as follows.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ d & -b_2 & -b_3 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -d & -b_2 & -b_3 \end{pmatrix},$$
$$A_{u1} = A_{u2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_2 & b_3 \end{pmatrix}.$$

The membership functions of the fuzzy sets are defined as

$$M_1(x(t)) = \frac{1}{2}\left(1 + \frac{x_1(t)}{d}\right), \; M_2(x(t)) = \frac{1}{2}\left(1 - \frac{x_1(t)}{d}\right) \tag{31}$$

So, one can gets $h_i(z(t))$ as follows.

$$h_1(z(t)) = M_1(z(t)) \times N_1(z(t)) \tag{32}$$

$$h_2(z(t)) = M_2(z(t)) \times N_2(z(t)) \tag{33}$$

where $N_1(z(t)) = N_2(z(t)) = 1$.

Now, according to the designed adaptive control laws (16)–(18), we have

$$u(t) = \hat{A}_{u1}x(t) + K_1x(t) + \hat{A}_{u2}x(t) + K_2x(t) \tag{34}$$

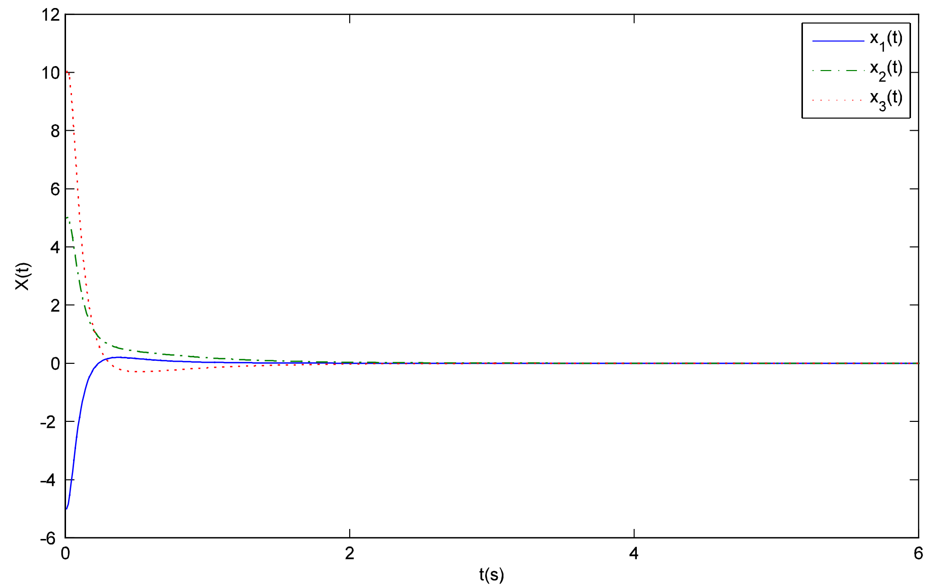

Fig. 1 State trajectory of the controlled Genesio–Tesi system

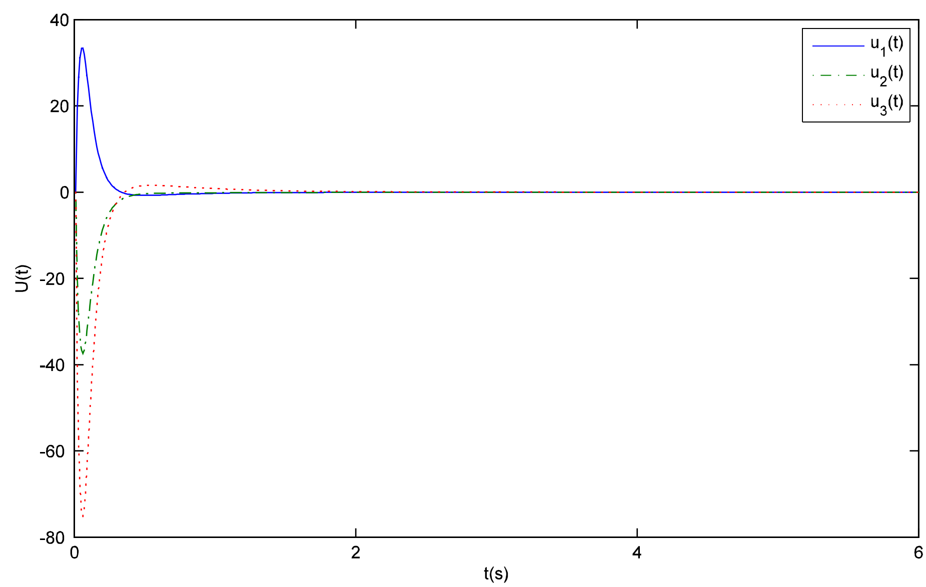

Fig. 2 Time response of the control inputs applied to the Genesio–Tesi system

where $K_1 = K_2 = diag(2,2)$ are chosen in the simulation and $\hat{A}_{ui}$ is an estimation of $A_{ui}(i = 1, 2)$ which is adapted as follows.

$$D^{\alpha}\hat{A}_{ui} = \sigma_{A_{ui}} \cdot h_i(z(t)) \cdot x(t) \cdot x^T(t),\ \hat{A}_{ui}(0) = 0 \tag{35}$$

where $\delta_{A_{ui}} = 2$ is assumed in the simulation.

The initial values of the system are assigned as $x_1(0) = -5$, $x_2(0) = 5$ and $x_3(0) = 10$. The state trajectory of the fractional-order Genesio–Tesi system controlled by the adaptive controller (34) is depicted in Fig. 1. As it can be seen, the states of the system converge to zero. This means that zero equilibrium point of the system is effectively stabilized. The time history of the control input (34) is plotted in Fig. 2. As it is shown, the control signals are bounded. Thus, the proposed controller can be realized in practical situations. Figures 3 and 4 depict the time responses of the adaption parameters $\hat{A}_{u1}$ and $\hat{A}_{u2}$. It is clear that all the adaption parameters attain some constants which implies that they are bounded and that the internal stability of the system is guaranteed.

In order to compare the performance of the proposed attenuation strategy to the other well-known existing methods in the literature, the method which has been proposed in reference [15] is taken into account. As Fig. 5 shows, state trajectories of the proposed technique in [15] contain

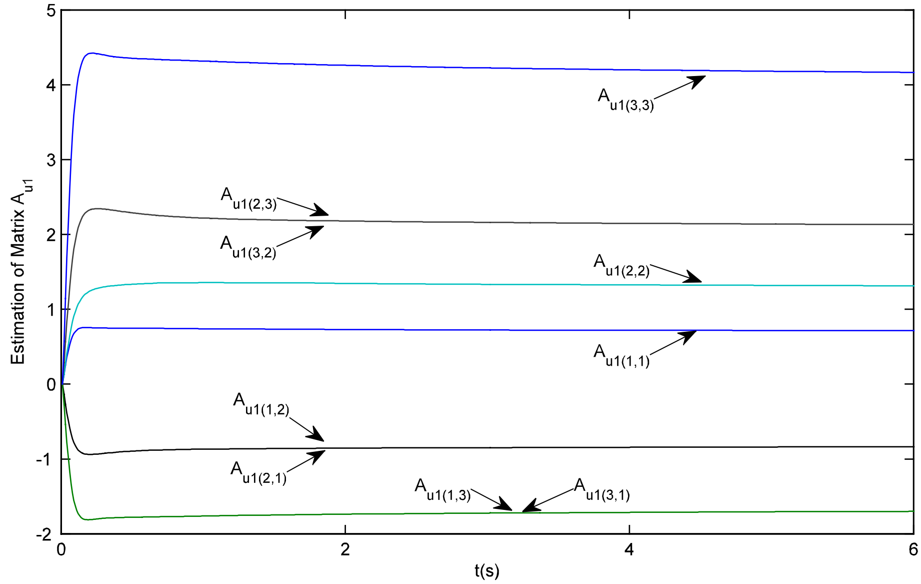

Fig. 3 Estimated parameters for matrix A_{u1} of the Genesio–Tesi system

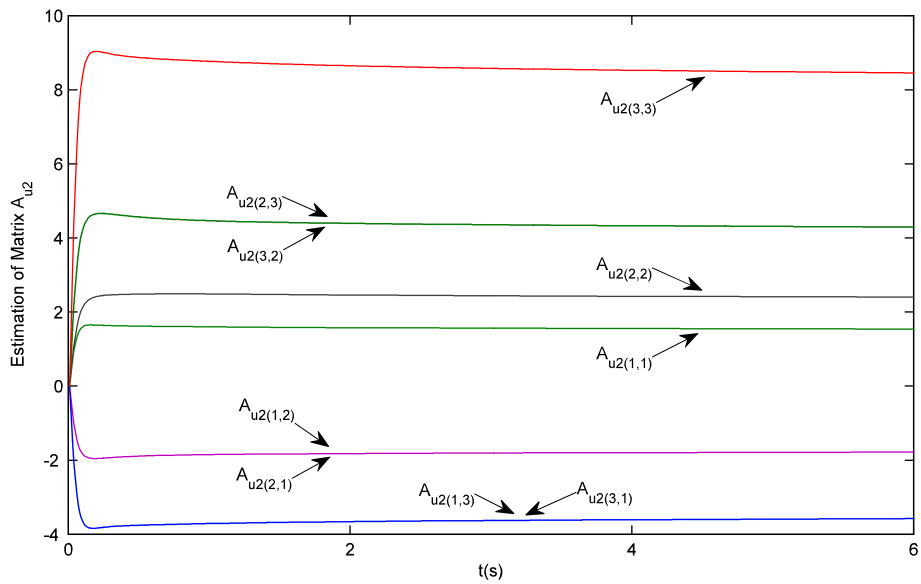

Fig. 4 Estimated parameters for matrix A_{u2} of the Genesi–Tesi system

steady state errors. But, our method does not suffer from such permanent oscillations. Thus, it can be concluded that the proposed method in this paper is more efficient and practical compared to that in [15].

4.2 Control of non-autonomous fractional-order horizontal platform system

Consider a non-autonomous fractional-order horizontal platform systems (FOHPS) whose dynamics is governed by [19]

$$\begin{cases} D^{\alpha}x_1(t) = x_2(t) \\ D^{\alpha}x_2(t) = -ax_2(t) - b\sin x_1(t) + l\cos x_1(t)\sin x_1(t) + h\cos \omega t \end{cases} \tag{36}$$

where $a = {}^4/_3, b = 3.776, l = 4.6 \times 10^{-6}, h = \frac{3.4}{4}$ and $\omega = 1.8$ are the parameters of the FOHPS system and α is equal 0.8 to ensure an oscillatory for the *FOHPS* [19]. It should be noted that the *FOHPS* system (36) may show stable behaviors for the different values of α. So, the stabilization problem will be of no sense for such stable cases. It is also noticed that according to the simulation tries, the other values of α which make the system behavior to be chaotic, have no effects on the controller performance.

Now, for applying the designed adaptive control scheme, the T–S fuzzy model representation of the FOHPS (36) is obtained first. By defining four fuzzy sets, one can achieve the following fuzzy system that shows the nonlinear equation of the FOHPS (36) by assumption of $x_1 \in (-d, d)$ and $d = 20$.

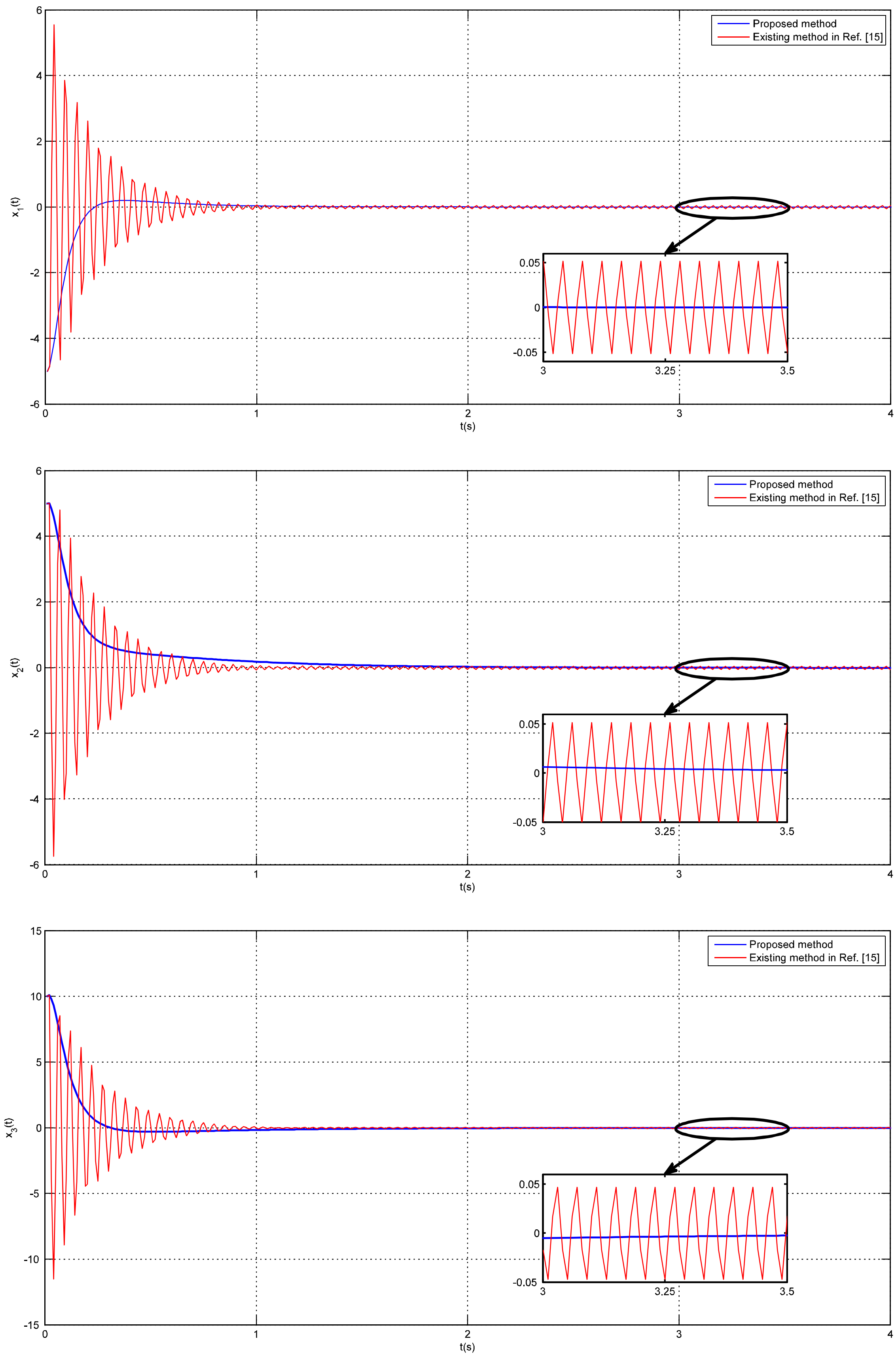

Fig. 5 States of the controlled Genesio–Tesi system with comparison to those of obtained by the method in [15]

Rule 1 If $x_1(t)$ is $M_1(x(t))$ Then $D^\alpha x(t) = A_1 x(t) + A_{u1} x(t) + B_1 \zeta(t) + B_{u1} \zeta(t) + u_1(t)$.

Rule 2 If $x_1(t)$ is $M_2(x(t))$ Then $D^\alpha x(t) = A_2 x(t) + A_{u2} x(t) + B_2 \zeta(t) + B_{u2} \zeta(t) + u_2(t)$.

Rule 3 If $x_1(t)$ is $N_1(x(t))$ Then $D^\alpha x(t) = A_3 x(t) + A_{u3} x(t) + B_3 \zeta(t) + B_{u3} \zeta(t) + u_3(t)$.

Rule 4 If $x_1(t)$ is $N_2(x(t))$ Then $D^\alpha x(t) = A_4 x(t) + A_{u4} x(t) + B_4 \zeta(t) + B_{u4} \zeta(t) + u_4(t)$. where $x(t) = [x_1(t), x_2(t)]^T$, $\varsigma(t) = [0, \cos(1.8t)]^T$ and A_i, A_{ui}, B_i and B_{ui} are parameters of the system to be supposed unknown.

Also, the membership functions of fuzzy sets are as follows.

$$M_1(x(t)) = \frac{1}{2} + \frac{10}{7} \sin\left(x_1(t)\right) \tag{37}$$

$$M_2(x(t)) = \frac{1}{2} - \frac{10}{7} \sin\left(x_1(t)\right) \tag{38}$$

$$N_1(x(t)) = \frac{1}{2} + \frac{50}{33} \sin\left(x_1(t)\right) \cos\left(x_1(t)\right) \tag{39}$$

$$N_2(x(t)) = \frac{1}{2} - \frac{50}{33} \sin\left(x_1(t)\right) \cos\left(x_1(t)\right) \tag{40}$$

So, one can get $h_i(z(t))$ as follows.

$$h_1(z(t)) = M_1(x(t)) \times N_1(x(t)) \tag{41}$$

$$h_2(z(t)) = M_1(x(t)) \times N_2(x(t)) \tag{42}$$

$$h_3(z(t)) = M_2(x(t)) \times N_1(x(t)) \tag{43}$$

$$h_4(z(t)) = M_2(x(t)) \times N_2(x(t)) \tag{44}$$

According to the proposed adaptive controller (16)–(18), one has

$$u(t) = -\sum_{i=1}^{4} h_i(z(t)) \left\{ \hat{A}_{ui} x(t) + \hat{B}_{ui} \zeta(t) + K_i x(t) + L_i \zeta(t) \right\} \tag{45}$$

where $K_1 = K_2 = K_3 = K_4 = \begin{pmatrix} 15 & 0 \\ 0 & 15 \end{pmatrix}$, $L_1 = L_2 = \begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix}$ and $L_3 = L_4 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ are chosen to satisfy the conditions of Eq. (27). Moreover, $\hat{A}_{ui}$ and $\hat{B}_{ui}$ are estimations of A_i and B_i, $i = 1, 2, 3, 4$ respectively, which are adapted as follows.

$$D^\alpha \hat{A}_{ui} = \sigma_{A_{ui}} \cdot \mathrm{h}_i(z(t)) \cdot x(t) \cdot x^T(t),\ \hat{A}_{ui}(0) = 0$$

$$D^\alpha \hat{B}_{ui} = \sigma_{B_{ui}} \cdot \mathrm{h}_i(z(t)) \cdot x(t) \cdot \zeta^T(t),\ \hat{B}_{ui}(0) = 0, \quad i = 1, 2, 3, 4$$

The initial values of the FOHPS (36) are chosen as $x_1(t) = 5$, $x_2(t) = -3$, $\sigma_{A_{ui}} = 10.5$and $\sigma_{B_{ui}} = 15.5$. The state trajectories of the system along the controller (45) are illustrated in Fig. 6. As it can be seen, the states of the system attain zero, which indicates that the oscillations of the original system are indeed suppressed and the origin equilibrium point is asymptotically stabilized. The time history of the control input (34) is appeared in Fig. 7. One can see that the control signals converge to fixed values. Moreover, Figs. 8 and 9 plot the time responses of the adaptive matrixes $\hat{A}_{ui}$ and $\hat{B}_{ui}$, $i = 1, 2, 3, 4$, respectively. Obviously, the adaptive parameters approach to bounded values, which implies that the auxiliary adaptive variables are bounded and the controller is feasible in practice.

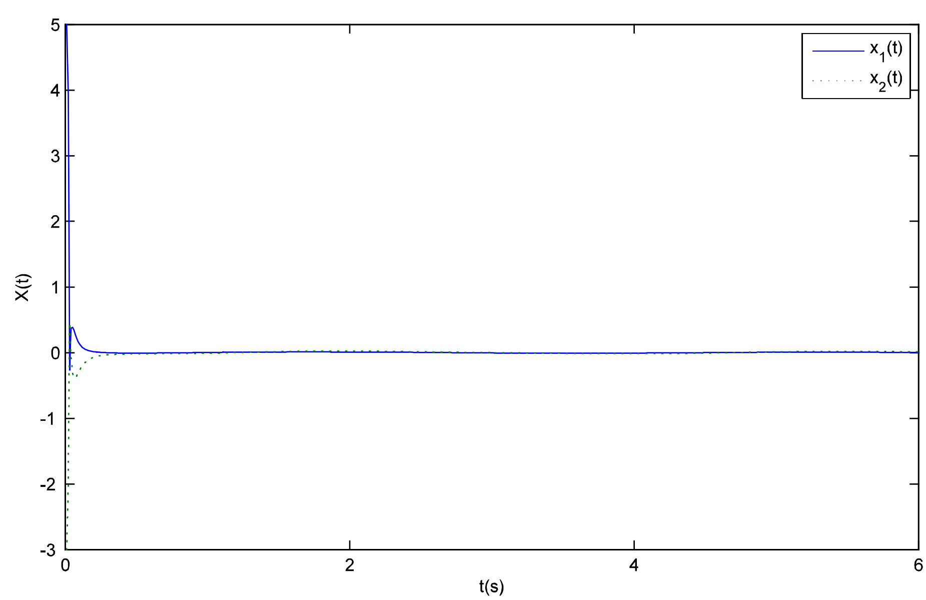

Fig. 6 State trajectories of the controlled FOHPS system

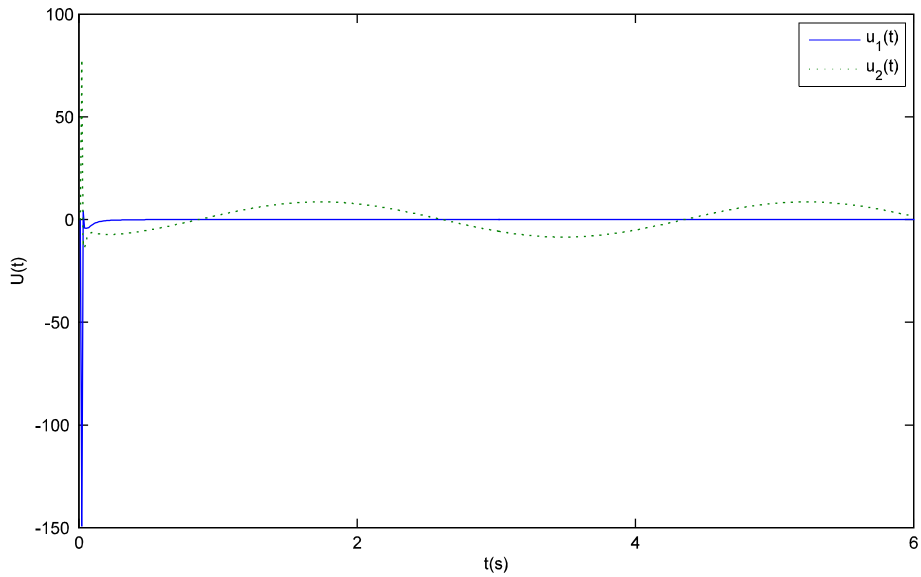

Fig. 7 Time response of the control inputs applied to the FOHPS system

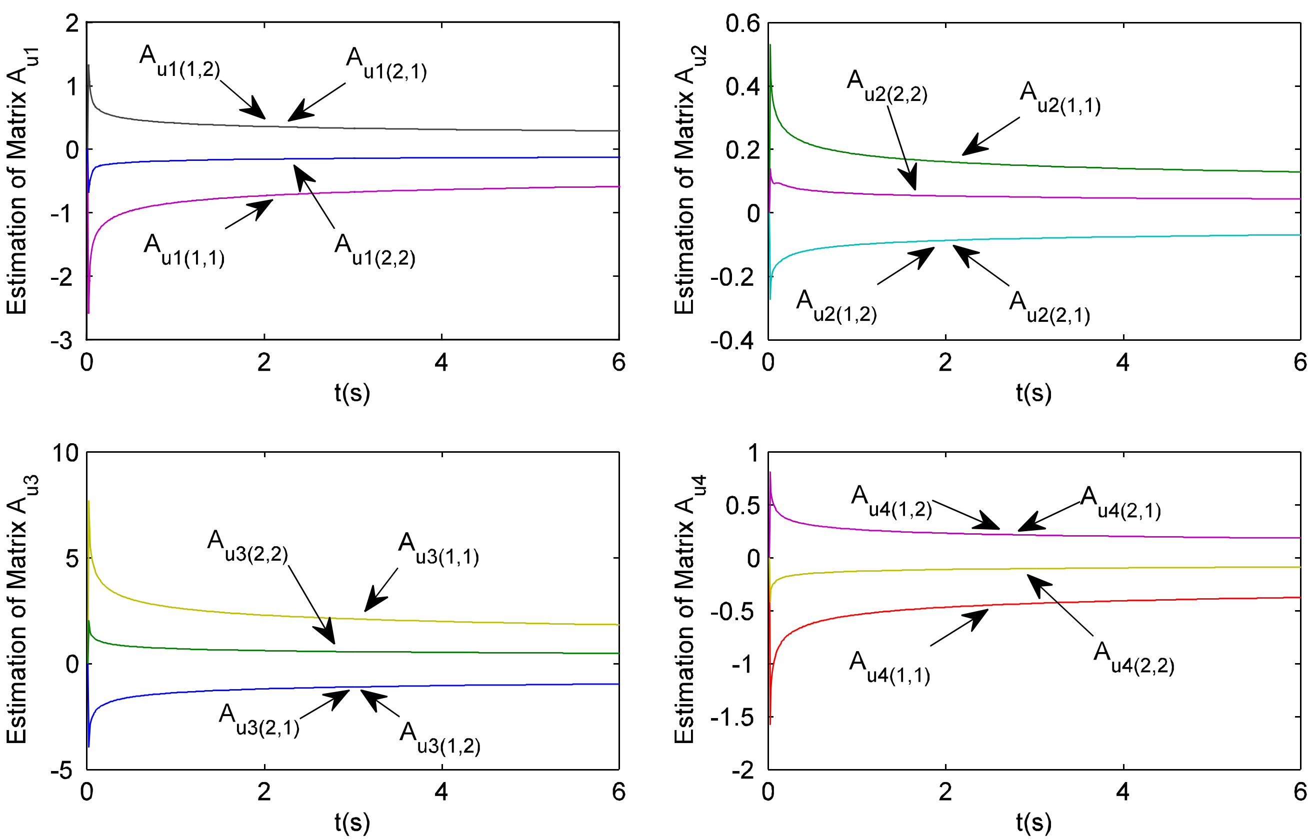

Fig. 8 Estimated parameters for the matrix A_{ui} of the FOHPS system

5 Conclusions

In this paper, the problem of control and stabilization of uncertain non-autonomous fractional-order systems is investigated. First, the intelligent Takagi–Sugeno (T–S) fuzzy models with if-then rules are constructed to represent the system dynamics. Then, an adaptive approach is adopted to estimate the unknown parameters and uncertainties of the system. Subsequently, based on the T–S fuzzy control technique and Lyapunov's stability theorem, linear-like

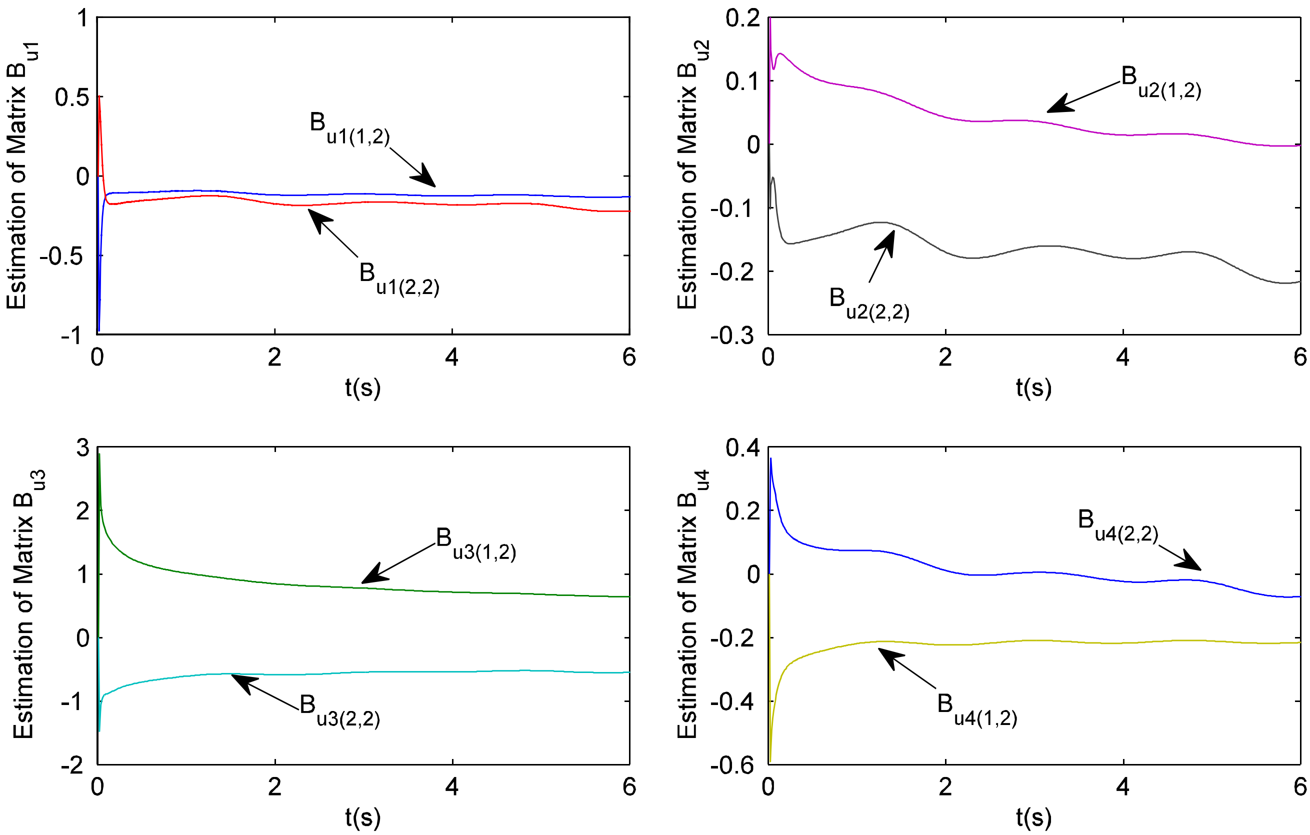

Fig. 9 Estimated parameters for matrix B_{ui} of the FOHPS system

control rules associated with some gain matrices provided to ensure that the system states will approach to zero as time goes infinite. After that, the developed controller is applied for stabilization of a large class of fractional-order non-autonomous systems. Two numerical examples are also presented to validate the analytical results of the article and to illustrate that the designed adaptive schemes are feasible in real world applications. It is worth to note that the results of this paper can be applied for control of real fractional-order systems, such as fractional-order electrical circuits and mechatronic devices, in spite of having limited knowledge about the time-variant and time-invariant parameters of the system. Extending the results of this paper for design of T–S fuzzy controllers for fractional-order systems with input saturation remains as the future work of the authors.

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