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Analytical properties of an imperfect repair model and application in preventive maintenance scheduling

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ABSTRACT

The paper considers repairable systems under imperfect repair. The failure rate of a new system is assumed to follow a Weibull distribution and the repair efficiency is characterized by a Kijima type II virtual age model named Arithmetic Reduction of Age with infinite memory. An analytical approach to obtain the distribution of the inter-failure times is presented. The existence of a stationary regime is highlighted and the limiting distributions are explicitly derived. In this context, an optimal age-based preventive maintenance policy can be implemented. Three approaches are proposed, considering a static, a dynamic or a failure limit policy. Numerical simulations are presented to illustrate the policies.

1. Introduction

During their operational life, industrial systems are subject to repair actions when a failure occurs. A repair activity is aimed to reduce the failure rate of the system and to extend its useful lifetime. The maintenance process has to take into account both the intrinsic aging of the system and the repair effectiveness. These two elements allow a better understanding of the system behavior in the short and long terms and the maintenance policy can be adapted consequently.

Repair efficiencies are commonly assumed to be either minimal or perfect. A minimal or As Bad As Old (ABAO) repair assumes that the system is restored to its operational condition just before the failure. A perfect or As Good As New (AGAN) repair consists in restoring the system to a new and identical one. Minimal repair and perfect repair can be characterized by a non-homogeneous Poisson process and a renewal p rocess (Ascher & Feingold, 1984), respectively. However, for a repairable system, these assumptions are not always realistic as the system can be effectively repaired but is not renewed. This situation is described as imperfect maintenance (Pham & Wang, 1996). A thorough account of imperfect maintenance modeling for repairable systems is developed by Lindqvist (2006). In the context of imperfect repair, the implementation of optimal maintenance policies have been developed by Nakagawa (2005) and by Pham and Wang (2006). The optimization of imperfect maintenance policies are discussed considering reliability block diagrams (Levitin & Lisnianski, 2000) and examples of application to failure data are presented by Baker (2001) and Dijoux and Gaudoin (2014).

Virtual age models (Kijima, Morimura, & Suzuki, 1988) are the most frequently used imperfect repair models. The principle is that the wear-out does not depend on the chronological age of the system, but on a virtual age, commonly between zero and the elapsed time since the system was new. A virtual age model is entirely characterized by the failure rate of a new system and by the virtual age assumptions. In particular, Kijima (1989) has proposed two widespread classes of virtual age assumptions. He supposes that each repair efficiency is represented by a random variable supported on the interval [0,1]. A model under Kijima Type I assumption is such that a repair rejuvenates the virtual age of a proportional amount of the last inter-failure duration, whereas a model under Kijima Type II assumption supposes that the rejuvenated amount is proportional to the virtual age just before the repair. A particular case is to consider that the repair efficiency is a constant $\rho \in [0, 1]$, called restoration factor. The resulting models have been developed by Malik (1979) and by Brown, Mahoney, and Sivazlian (1983) for the Kijima type I and II models, respectively. A unified version of the last two models has been presented by Doyen and Gaudoin (2004), called model of arithmetic reduction of age with memory m, and denoted ARA_m. The ARA_1 and ARA_∞ models are special cases of the Kijima Type I and II models, respectively. Theoretical results on the ARA_1 model are developed in the literature (Kijima & Sumita, 1986; Malik, 1979; Yevkin, 2012) and are applied

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in maintenance scheduling (Dimitrakos & Kyriakidis, 2007; Jiang, Makis, & Jardine, 2001; Kijima et al., 1988; Love, Zhang, Zitron, & Guo, 2000; Makis & Jardine, 1993). Similarly, some properties of the ARA_∞ model are discussed when it is introduced (Doyen & Gaudoin, 2004; Kijima, 1989). These models are also developed in the presence of different kinds of maintenance actions (Dijoux & Idée, 2013; Doyen & Gaudoin, 2011).

Last and Szekli (1998) have proven the convergence of the Kijima Type II model, and hence of the ARA_∞ model, to a steady-state regime. Finkelstein (2008) has proven the convergence of the ARA_∞ model to a steady-state regime when the repair efficiency depends on the chronological age of the system. In contrast, Doyen (2010) has proven that the ARA_1 model behaves asymptotically as a non-homogeneous Poisson process. These properties highlight a major difference between the ARA_1 and ARA_∞ models when the restoration factor is in the interval]0, 1[. If the failure rate of a new system is increasing monotonically to infinity, the inter-failure times converge to zero for the ARA_1 model and to a stationary distribution for the ARA_∞ model.

This paper aims to solve a maintenance problem inspired by a failure data set of electrical transformers given by French electrical company (Électricité de France-EDF). Data consist of maintenance dates without any information on type of maintenance and the age of the system. The data concern systems running for a long and unknown period of time. However, the failure data are available on a short and recent time window even if the systems have been implemented decades ago. The data could be assumed to correspond to the system stable regime, but one needs appropriate tools to take into account the lack of information on the system history. In this framework, it is of essential interest to infer the system's behavior in order to improve the maintenance policy and to plan efficient maintenance operations. Since the ARA_∞ model exhibits a convergence property (existence of a stable regime), it is a suitable candidate to model the data set.

In fact the ARA models, in particular ARA_1 and ARA_∞ have been intensively studied and in this framework many preventive maintenance policies are proposed. Nevertheless, in the literature, the imperfect preventive maintenance in an infinite horizon is rarely addressed. For instance

- Dagpunar (1998) proposes block replacement policy with ARA_∞ corrective maintenance and perfect preventive maintenance. The optimal duration between two preventive maintenance is derived, based on the computation of mean number of corrective maintenance actions during this period.
- Kijima et al. (1988) study block replacement policy with ARA_1 corrective maintenance and perfect preventive maintenance. They develop an approach to compute the mean number of ARA_1 corrective maintenance between two consecutive perfect preventive maintenance actions.
- Gilardoni, de Toledo, Freitas, and Colosimo (2015) study periodic preventive maintenance policy using ARA_1corrective maintenance and perfect preventive maintenance. The policy is first built for finite horizon and then extended to an infinite horizon.
- Tsai, Liu, and Lio (2011) study planned preventive maintenance policy. The preventive maintenance effect is similar to the ARA_1 maintenance, whereas corrective maintenance is minimal. The policy is optimized in a finite horizon.
- Gilardoni and Colosimo (2007) propose a similar policy (ARA_1 preventive maintenance and minimal corrective maintenance) optimized on an infinite horizon.

The main contribution of our paper is first to derive original theoretical properties of the $WARA_\infty$ model and then to develop original preventive maintenance policies. Regarding specifically the optimization of the preventive maintenance policy, our main contributions are listed as follows:

- For the first time, maintenance policies are derived on an infinite horizon considering imperfect CM and imperfect PM. Therefore there is at no moment an as good as new (perfect, renewal) replacement of the system during its operational lifetime. All the papers in the literature consider renewals under an infinite horizon.
- Thanks to the theoretical results in the stationary regime from the first Section, efficient and analytical approximations of the optimal maintenance policy are obtained without using Monte Carlo simulations.

The remainder of the paper is organized as follows. In Section 2, properties of the $WARA_\infty$ are developed to obtain statistical distributions of interest in both the transient and the stationary regimes. Based on these distributions, planned preventive maintenance policies are proposed in Section 3, along with numerical illustrations.

2. Properties of the $WARA_\infty$ model

2.1. The repair process

A repairable system has been observed since it was new. The observations consist of the successive maintenance times $\{T_i\}_{i\geq 0}$. The corresponding inter-maintenance times are denoted $\{X_i\}_{i\geq 1}$ and the repair process can also be characterized by a counting process $\{N_t\}_{t\geq 0}$ where $N_t = \sum_{i=1}^{\infty} \mathbb{1}_{\{T_i<t\}}$. By convention, T_0 and X_0 are equal to zero and time can be either calendar or operational. The distributions are obtained from the failure intensity defined in (1), where $\mathcal{H}_{t^-}$ is the history of the repair process at time t^-, commonly the failure times before t.

$$\forall t \geq 0,\ \lambda_t = \lim_{\Delta t\to 0} \frac{1}{\Delta t} P(N_{t+\Delta t} - N_{t^-} = 1|\mathcal{H}_{t^-}) \tag{1}$$

The failure rate of a new system, called *initial failure intensity* and denoted $\lambda(t)$, is assumed to be a deterministic and continuous function of time. It corresponds to the hazard rate of T_1. The cumulative hazard rate function is denoted $\Lambda(t) = \int_0^t \lambda(u)du$. f, F and R are the corresponding probability density function, cumulative distribution function and survival function, respectively. As industrial systems are assumed to wear out, the initial failure intensity is traditionally increasing. Consequently, the two-parameter Weibull distribution has been chosen as in (2). For wearing-out systems, the shape parameter β is greater than 1.

$$\forall t \geq 0,\ \lambda(t) = \alpha\beta t^{\beta-1} \tag{2}$$

A virtual age model (Kijima et al., 1988) assumes that after the *ith* repair, the system behaves as a new and unmaintained one of age A_i. This age is called *effective age*. The assumption is mathematically described in (3), where Z is the time to failure of a new system and has the same distrib ution as X_1.

$$\forall i \geq 0, \forall t \geq 0,\ P(X_{i+1} > t|X_1, \ldots, X_i) = P(Z > A_i + t|Z > A_i) \tag{3}$$

The conditional survival function of the $(i+1)th$ inter-failure time in (3) is simply $R(A_i + t)/R(A_i)$. The age of the system A_0 at the beginning of the observation is zero if the system is as good as new, and greater than 0 otherwise. At a given time t, the virtual age of the system V_t is obtained from the latest effective age and the elapsed time since the last repair as in (4).

$$V_t = A_{N_{t^-}} + t - T_{N_{t^-}} \tag{4}$$

The virtual age of the system just before the *ith* repair is denoted A_i^-. The failure intensity can be derived from the initial failure intensity as in (5). The variation of the virtual age V_t and of the chronological time are identical between two consecutive failures.

Dorado, Hollander, and Sethuraman (1997) have presented an extension of the virtual ages where this variation can be accelerated or decelerated.

$$\lambda_t = \lambda(V_t) = \lambda(A_{N_{t-}} + t - T_{N_{t-}}) \tag{5}$$

Many models have been developed from different assumptions on the virtual ages (Brown & Proschan, 1983; Dijoux & Idée, 2013; Doyen & Gaudoin, 2004; Kijima, 1989). In particular, a Kijima type II model (Kijima, 1989) assumes that regarding the ith repair, the effective age A_i is proportional to A_i^-. The model of arithmetic reduction of age with infinite memory ARA_∞ (Doyen & Gaudoin, 2004) assumes that this proportion is a constant ρ in the interval [0,1] as in (6). ρ is called the repair efficiency or the restoration factor.

$$A_i = (1-\rho)A_i^- = (1-\rho)(A_{i-1} + X_i) \tag{6}$$

In particular, a minimal and a perfect maintenance can be modeled with $\rho = 0$ and $\rho = 1$, respectively. The effective age in (6) can be explicitly expressed in terms of the inter-failure times and A_0, as in (7). One can find useful applications of the ARA_∞ model in different situations (Bartholomew-Biggs, Ming, & Xiaohu, 2009; Brown et al., 1983; Clavareau & Labeau, 2009; Dijoux & Idée, 2013; Kahle, 2007; Yun & Choung, 1999).

$$A_i = (1-\rho)^i A_0 + \sum_{j=1}^{i} (1-\rho)^{(i-j+1)} X_j \tag{7}$$

Combining (5) and (7), a model under the ARA_∞ assumption is defined by the initial age A_0, the initial failure intensity $\lambda(.)$ and the repair efficiency ρ. The Weibull-ARA_∞ model , denoted $WARA_\infty$, consists of the ARA_∞ assumption, the Weibull initial failure defined in (2) and a null initial age.

Even if it seems very simple, the constant restoration factor is widely used in the literature and can describe the maintenance effect in many real industrial cases. For instance Malik (1979) uses the constant restoration factor to describe the effect of planned maintenance on system's virtual age. Guo and Love (1992) also use a constant restoration factor to build their imperfect maintenance model. One can enumerate several other papers using the same assumption to quantify the maintenance effect (Bartholomew-Biggs et al., 2009; Love et al., 2000; Yevkin, 2012; Yevkin & Krivtsov, 2013). Furthermore, when maintenances are well in place, the maintenance activities are relatively homogeneous and the constant restoration factor seems appropriate. In practice, the number of observations is usually not large during the analysis of a maintenance data set. It is consequently necessary to find a compromise between the number of parameters of the model and a high-level of precision on the system. Considering a system under one kind of maintenance only, the use of one parameter for the maintenance efficiency allows to obtain a model which is flexible enough and which approximately takes into account the global maintenance efficiency on a system level. Naturally, in presence of multiple types of maintenances (preventive, corrective associated with different failure modes), a more general modeling can be proposed as in Doyen and Gaudoin (2006) and Lindqvist (2006), but this approach is not considered in this paper.

2.2. Analytical developments of the $WARA_\infty$ model

In the following, properties of the $WARA_\infty$ model are derived. The restoration factor ρ is assumed to be in the open interval (0, 1). Properties of the Weibull distributions have been developed in the case of a renewal process ($\rho = 1$) (Lomnicki, 1966; Yannaros, 1994) and in the case of a non-homogeneous Poisson processes ($\rho = 0$) (Crow, 1974; Rigdon & Basu, 2000). The marginal distributions of A_n, X_n and A_n^- are derived in the transient and the stationary regimes of the system.

Proposition 1.

$$\forall n \geq 1,\ A_n = (1-\rho)\Lambda^{-1}(\Lambda(A_{n-1}) + \xi_n)$$

where Λ^{-1} is the inverse function of Λ and ξ_n has an exponential distribution with rate 1 and is independent of the previous random variables $\{\xi_j\}_{j=1,\dots,n-1}$.

Proposition 1 remains valid for other initial failure intensities, given that the inverse function Λ^{-1} can be defined. The proof is a straightforward application of the standard method of simulation by inversion of the cumulative distribution function. The details of the proof can be found in Appendix A. In the case of the $WARA_\infty$ model, $\Lambda^{-1}(t) = (t/\alpha)^{1/\beta}$ and the effective ages can be expressed as in (8).

$$\forall n \geq 1,\ A_n = (1-\rho)\alpha^{-\frac{1}{\beta}}\left(\alpha A_{n-1}^{\beta} + \xi_n\right)^{\frac{1}{\beta}} \tag{8}$$

Proposition 2.

$$\forall n \geq 1,\ A_n = (1-\rho)\alpha^{-\frac{1}{\beta}}\left(\sum_{i=1}^{n}(1-\rho)^{\beta(n-i)}\xi_i\right)^{\frac{1}{\beta}}$$

in the case of the $WARA_\infty$ model, where $\{\xi_i\}_{i=1..n}$ are n independent exponential random variables with rate 1.

Proposition 2 is a direct consequence of Proposition 1. A proof by induction is derived in Appendix B. Multiplying an exponential random variable by a positive constant preserves the exponential nature of the distribution. The summand in the expression of the effective ages in Proposition 2 is therefore exponentially distributed. A sum of independent exponential random variables follows a hypo-exponential distribution (Gaudoin & Ledoux, 2007). The hypo-exponential distribution is a special case of the phase-type distribution (Slud & Suntornchost, 2014). The effective ages A_n can be regarded as a transformation from a hypo-exponential distribution. Let us introduce in (9) the q-Pochhammer operator, also called q-shifted factorial (Mcintosh, 1999), which is a basic hypergeometric series (Gasper & Rahman, 1990).

$$\text{(q-shifted factorial)}\quad (a, q)_k = \prod_{j=0}^{k-1}(1 - aq^j) \tag{9}$$

Proposition 3. *Denoting $q = (1-\rho)^\beta$, the survival distribution of A_n in the $WARA_\infty$ model is:*

$$\forall n \geq 1,\ R_{A_n}(t) = \sum_{k=1}^{n}\frac{1}{(q,q)_{n-k}(\frac{1}{q},\frac{1}{q})_{k-1}}e^{-\frac{\alpha t^\beta}{q^k}}$$

The expression of the survival distribution of A_n in Proposition 3 is derived from the survival function of a hypo-exponential distribution. The complete solution of the proof is derived in Appendix C.

Proposition 4. *The survival function of X_{n+1} in the $WARA_\infty$ model is given by:*

$$R_{X_{n+1}}(t) = \begin{cases} \sum_{k=1}^{n}\dfrac{\int_0^\infty \alpha\beta x^{\beta-1}e^{-\alpha(x+t)^\beta + \alpha(1-q^{-k})x^\beta}dx}{q^k(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}, & n \geq 1 \\ e^{-\alpha t^\beta}, & n = 0 \end{cases}$$

Given A_n, the conditional distribution of the consecutive inter-failure time X_{n+1} can be easily obtained. The distribution of X_{n+1} is therefore computed by a conditioning on A_n, when necessary. Additional details of the proof are presented in Appendix D.

Proposition 5. *By denoting the Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, the expectations of A_n and X_{n+1} in the $WARA_\infty$ model are:*

$$\forall n \geq 1,\ E[A_n] = \alpha^{-1/\beta}\Gamma\left(\frac{1}{\beta}+1\right)\sum_{k=1}^{n}\frac{q^{\frac{k}{\beta}}}{(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}$$

$$\forall n \geq 0,\ E[X_{n+1}] = \alpha^{-1/\beta}\Gamma\left(\frac{1}{\beta}+1\right)\sum_{k=1}^{n+1}\frac{q^{\frac{k-1}{\beta}}\left(1-q^{\frac{1}{\beta}}+q^{n+1-k+\frac{1}{\beta}}\right)}{(q,q)_{n+1-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}$$

The mean of both of the distributions are obtained by integrating the corresponding survival functions over the interval [0, ∞[. A proof is detailed in Appendix E. Another quantity of interest is the virtual age of the system just before a failure. The distribution of A_n^- is presented in Proposition 6. The proof follows directly from the relation between A_n and A_n^- in (6).

Proposition 6. *In the $WARA_\infty$ model, the distribution of the virtual age just before the nth repair A_n^- is given by:*

$$\forall n \geq 1,\ R_{A_n^-}(t) = \sum_{k=1}^{n}\frac{1}{(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}e^{-\frac{\alpha t^\beta}{q^{k-1}}}$$

The expectation of A_n^- has the following form:

$$\forall n \geq 1,\ E[A_n^-] = \alpha^{-1/\beta}\Gamma\left(\frac{1}{\beta}+1\right)\sum_{k=1}^{n}\frac{q^{\frac{k-1}{\beta}}}{(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}$$

In a Markovian approach, the effective ages A_n can be represented by a continuous-state discrete-time Markov chain. The initial condition and the transition probabilities are presented in Proposition 7. A proof is provided in Appendix F.

Proposition 7. *The effective ages A_n in the $WARA_\infty$ model can be characterized by a Markov chain on the continuous state space $\mathbb{R}_+$ as follows:*

$$A_0 = p(s,t) = \frac{-\partial P(A_{n+1} > s|A_n = t)}{\partial s} = \frac{\alpha\beta s^{\beta-1}}{(1-\rho)^\beta}e^{-\alpha((\frac{s}{1-\rho})^\beta - t^\beta)}\mathbb{1}_{\{s\geq(1-\rho)t\}}$$

It is already proven that a model based on the ARA_∞ assumption admits a stationary distribution under classical conditions on the initial failure intensity (Last & Szekli, 1998). In particular, the $WARA_\infty$ model converges to a steady-state regime. In Proposition 8, another evidence of the convergence can be obtained from Propositions 3 and 4. In addition, the limiting distributions of the effective ages and of the inter-failure times can be derived analytically. The details of the proof are provided in Appendix G.

Proposition 8. *In the $WARA_\infty$ model, the effective ages and the inter-failure times converge to a stationary regime. The corresponding limiting distributions A_∞ and X_∞ are characterized by their survival function and expectation as below.*

$$R_{A_\infty}(t) = \sum_{k=1}^{\infty}\frac{1}{(q,q)_\infty\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}e^{-\frac{\alpha t^\beta}{q^k}}$$

$$R_{X_\infty}(t) = \sum_{k=1}^{\infty}\left[\frac{1}{q^k(q,q)_\infty\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}\int_0^\infty \alpha\beta x^{\beta-1}e^{-\alpha(x+t)^\beta+\alpha(1-q^{-k})x^\beta}dx\right]$$

$$E[A_\infty] = \alpha^{-\frac{1}{\beta}}\Gamma\left(\frac{1}{\beta}+1\right)\sum_{k=1}^{\infty}\frac{q^{\frac{k}{\beta}}}{(q,q)_\infty\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}$$

$$E[X_\infty] = \alpha^{-\frac{1}{\beta}}\Gamma\left(\frac{1}{\beta}+1\right)\sum_{k=1}^{\infty}\frac{q^{\frac{k}{\beta}}(q^{-\frac{1}{\beta}}-1)}{(q,q)_\infty\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}$$

The probability distribution functions of A_1, A_2, A_3 and A_∞ for $\alpha = 1$, $\beta \in \{1.5, 2.5, 3.5\}$ and $\rho \in \{0.2, 0.4, 0.6\}$ are depicted in Figs. 1–3.

Furthermore,Finkelstein (2008) pointed out that if the distribution of the first inter-failure time X_1 is an IFR (Increasing Failure Rate) distribution, then the sequence of effective ages (A_1, A_2,... A_n,...) is monotonically increasing and the sequence of inter-failure times (X_1, X_2,... X_n,...) is monotonically decreasing. In the particular case of the $WARA_\infty$ model considered in this paper, since X_1 follows a IFR Weibull distribution, the sequence of effective ages (A_1, A_2,... A_n,...) is monotonically increasing. Therefore, $E[X_1] \geq E[X_2] \geq ... \geq E[X_\infty]$ which means in average the system is not getting younger under imperfect maintenance actions. However, after a long operation time and a large number of maintenance actions, the inter-failure times distribution will stabilize and in average they be equal to $E[X_\infty]$. Moreover, the probability distribution function of A_∞ is very similar to the distributions of A_1, A_2,... A_n, $\cdots$.

In the next section, the properties obtained for the $WARA_\infty$ model are used to implement optimal planned preventive maintenance policies.

3. Optimization of planned preventive maintenance policies

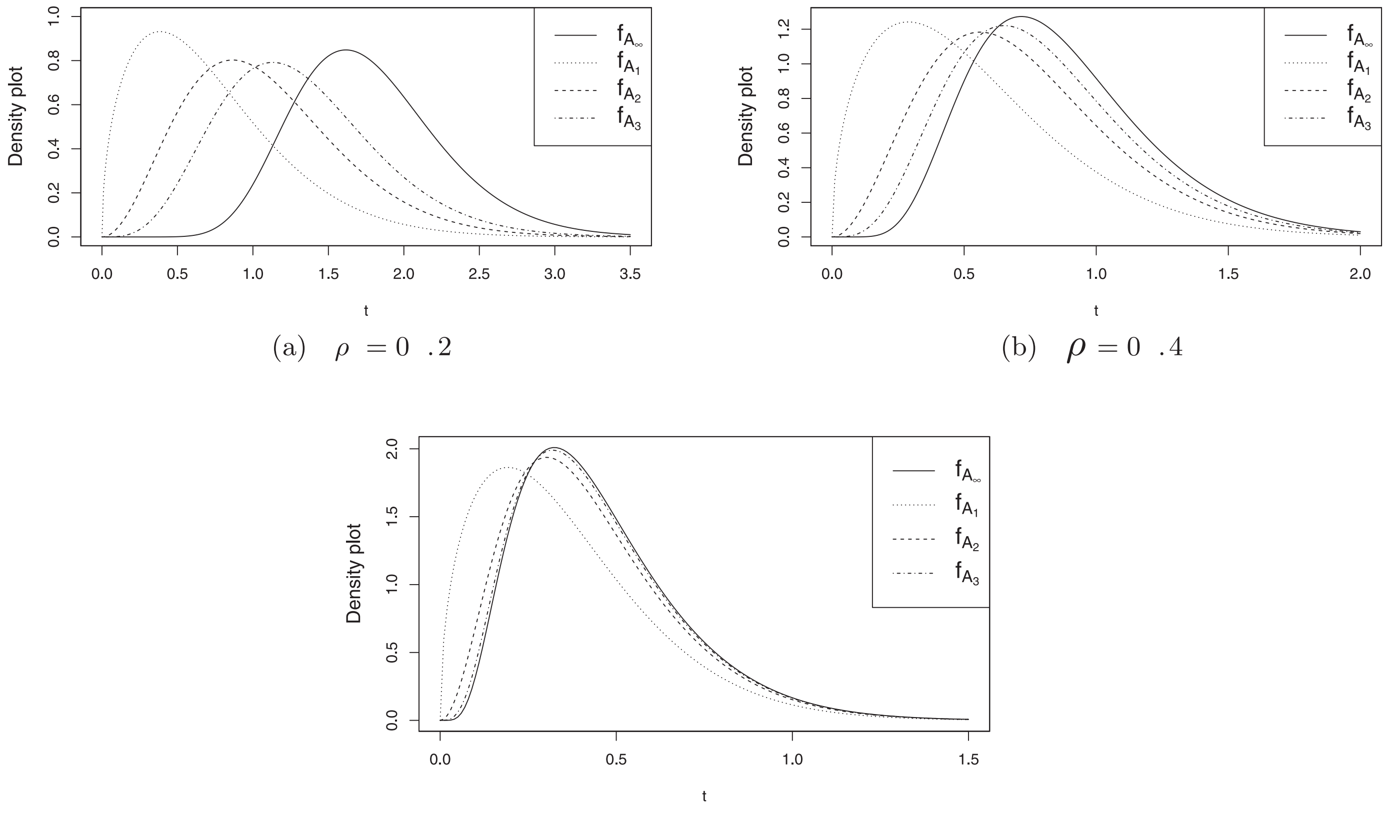
3.1. Preventive maintenance scheduling

One of the main objectives of failure data analysis is to predict possible future outcomes and to optimize maintenance actions accordingly. After the occurrence of multiple failures followed by a corrective maintenance, it is natural to consider the implementation of a preventive maintenance policy. A preventive maintenance can be either condition-based, performed according to the results from monitoring devices and from inspections or planned, performed at a scheduled time. The optimization of the policy can be obtained with regard to reliability, cost, availability and safety. Considering imperfect maintenance, condition-based preventive maintenance policies have been proposed for deteriorating systems when the degradation level of the system can be measured (Do & Bérenguer, 2012; Mercier & Castro, 2013; Nicolai, Frenk, & Dekker, 2009; Zhang, Gaudoin, & Xie, 2015). For lifetime distributions, imperfect preventive maintenance policies have also been developed (Jiang et al., 2001; Kijima et al., 1988; Nakagawa, 1980; Pham & Wang, 2006). In particular, under a Weibull initial failure intensity, optimal preventive maintenance strategies have been proposed using the ARA_1 (Toledo, 2014) and the ARA_∞ (Scarsini & Shaked, 2000; Yevkin & Krivtsov, 2013) assumption. The vast majority of these models assume that the system is replaced by a new one after either a replacement, a certain number of repairs or a period of time. This assumption is necessary if the inter-failure times converge to zero as for the majority of the Kijima type I models. However, for the Kijima type II models and in particular the $WARA_\infty$ model, the asymptotic behavior of the system allows not to have to replace the system by a new one at any given time. The proposed maintenance policies offer more flexibility to the ARA_∞ model in maintenance scheduling, especially for stable systems with previously unobserved failures.

In the following, based on the convergence properties of the $WARA_\infty$ model, three planned preventive maintenance policies are presented and their implementation are optimized based on maintenance costs.

3.2. Three planned preventive maintenance strategies for the $WARA_\infty$ model

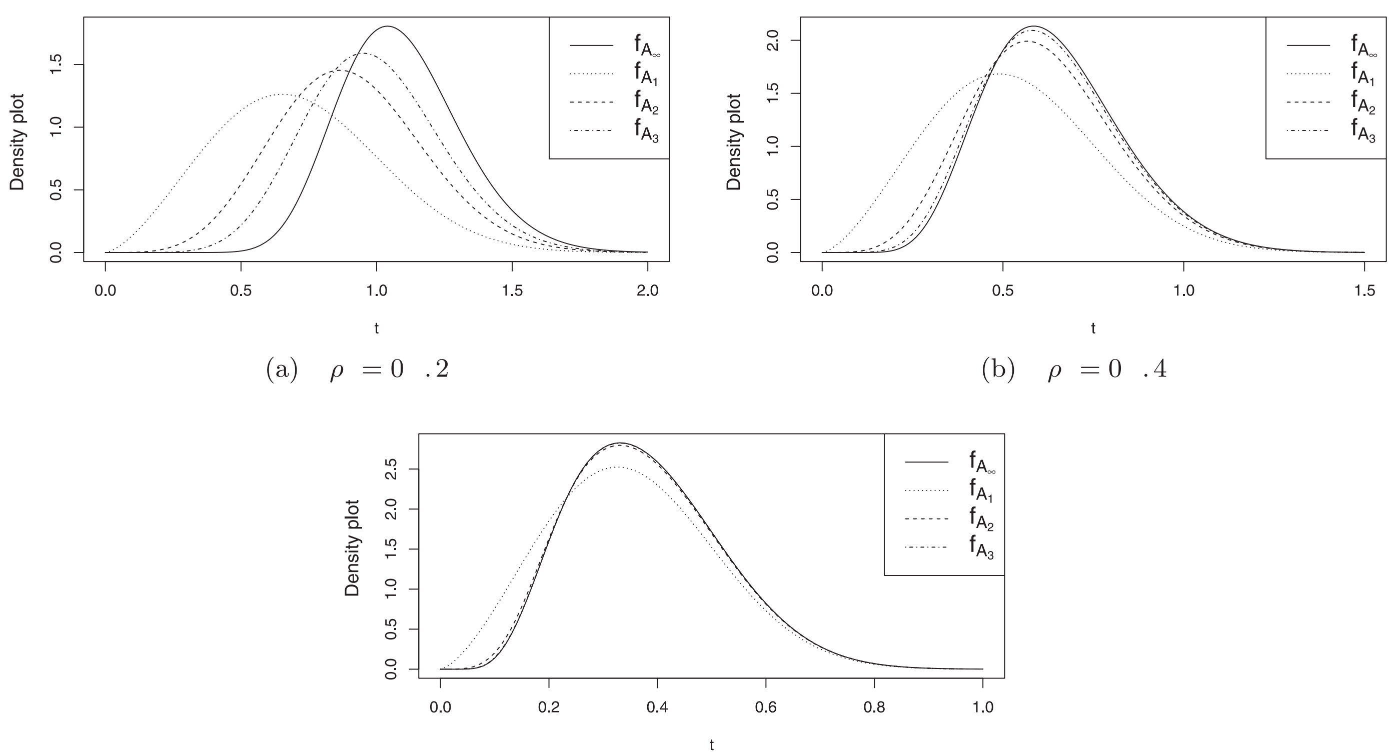
A repairable system which can be operated on an infinite horizon is considered. The assumption on the models are as follows:



(a) $\rho = 0.2$ (b) $\rho = 0.4$

(c) $\rho = 0.6$

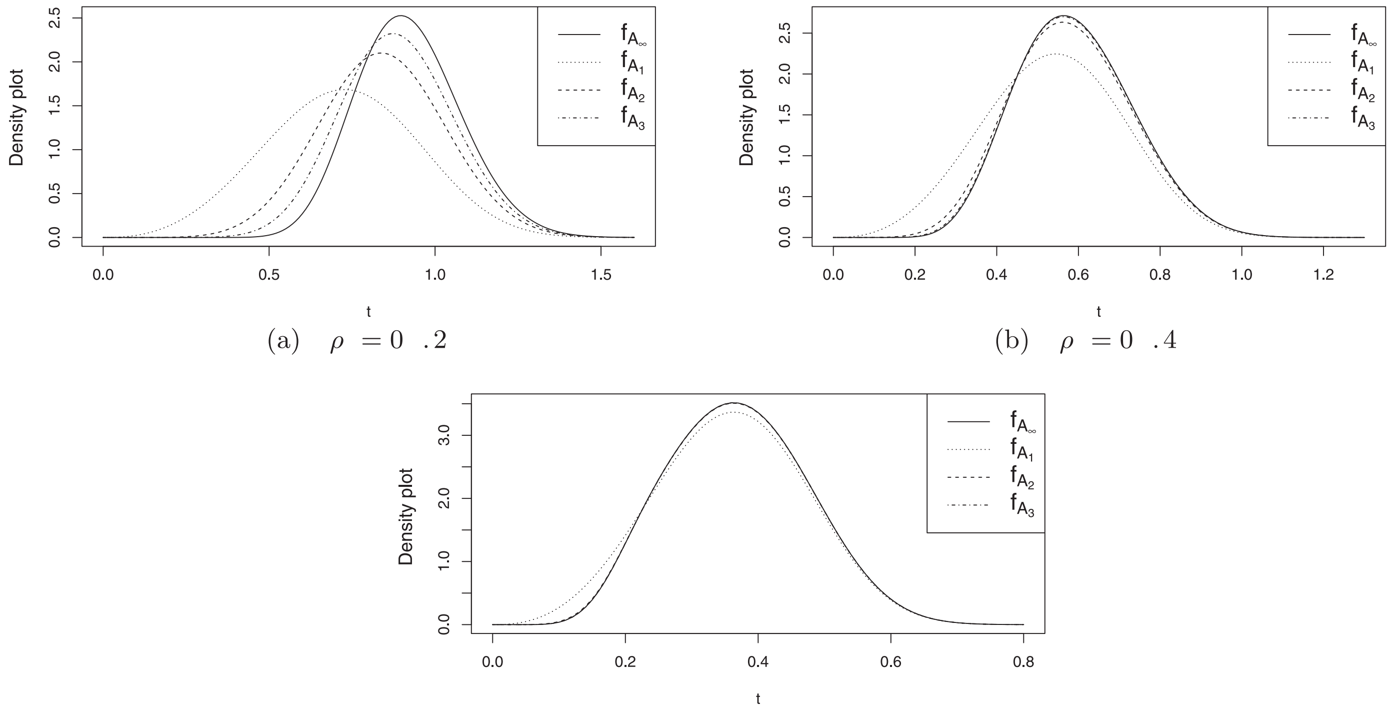
Fig. 1. Density plots with the setting $\alpha = 1$, $\beta = 1.5$, $\rho \in \{0.2, 0.4, 0.6\}$.



(a) $\rho = 0.2$ (b) $\rho = 0.4$

(c) $\rho = 0.6$

Fig. 2. Density plots with the setting $\alpha = 1$, $\beta = 2.5$, $\rho \in \{0.2, 0.4, 0.6\}$.



(a) $\rho = 0.2$ (b) $\rho = 0.4$ (c) $\rho = 0.6$

Fig. 3. Density plots with the setting $\alpha = 1, \beta = 3.5, \rho \in \{0.2, 0.4, 0.6\}$.

(i) At the beginning of the observation, the system is in the steady-state regime of the $WARA_\infty$ model with parameters (α, β, ρ) and the system has just been maintained. The time scale starts at zero and the same notation as in Section 2.1 is used.
(ii) The system is hereafter subjected to corrective maintenance (CM) and planned preventive maintenance (PM).
(iii) After the *ith* maintenance (PM or CM, $i \geq 0$), the system's virtual age is a_i and the duration δ^*_{i+1} to the next preventive maintenance is scheduled.
(iv) The maintenance process is characterized by a virtual age assumption on the maintenances and the intrinsic wear-out is a Weibull distribution with the same parameters (α, β).
(v) CM still follows a ARA_∞ assumption with parameter ρ.
(vi) PM follows a ARA_∞ assumption with the same restoration factor ρ.
(vii) The parameters of the model (α, β, ρ) are known.
(viii) Maintenance costs are known. Costs of CM and PM are C_c and C_p, respectively, with $C_c > C_p$. The cost of the *ith* maintenance $K(i)$ is obtained by comparing time-to-failures (10a) or virtual ages (10b).

$$K(i) = C_p + (C_c - C_p)\mathbb{1}_{\{X_i < \delta^*_i\}} \tag{10a}$$

$$K(i) = C_p + (C_c - C_p)\mathbb{1}_{\{a_{i-1} + X_i < a^*_i\}} \tag{10b}$$

At time t, the total cost $C(t)$ is simply $C(t) = \sum_{i=1}^{N_t} K(i)$. The performance of a policy is measured by its long-run average cost per unit of time as in (11).

$$C_\infty = \lim_{t \to \infty} \frac{C(t)}{t} \tag{11}$$

Different preventive maintenance strategies are possible, characterized by the choice of the durations $\{\delta^*_{i+1}\}_{i \geq 0}$. The three following preventive maintenance policies are developed:

- A static policy assumes that the durations δ^*_{i+1} are constant. The cost of one cycle is computed by comparing the actual time-to-failure and the proposed PM duration (10a). The optimal duration δ^* minimizes the cost function (11). δ^* is either obtained by Monte-Carlo simulations or approximated using the renewal theory and properties of Section 2.2.
- A dynamic policy computes the duration δ^*_{i+1} by taking into account the past of the maintenance process. The optimal duration δ^*_{i+1} minimizes a mean cost per unit time of the $(i+1)th$ cycle. The cost of one cycle is also computed with (10a).
- A failure limit policy defines an optimal virtual age s^* as a threshold not to exceed by the system's virtual age. The optimal value can be obtained by Monte-Carlo simulations. The policy compares the actual virtual age to the optimal value s^* to derive the next PM duration (10b).

Additional remarks on the assumptions of the model are discussed below.

- The vast majority of the research papers in preventive maintenance optimization assumes that the parameters of the models are known, for instances (Chien & Sheu, 2006; Nakagawa, 2005; Pham & Wang, 2006; Sandve & Aven, 1999). Even if the assumption that the parameters are known is simplistic in practice, it allows to identify the overall quality of a preventive maintenance strategy and provides general directions in terms of preventive maintenance optimization.
- As an additional PM strategy is adopted, the resulting model is not a $WARA_\infty$ model, unless $\delta^*_{i+1} = +\infty$. However, as all maintenances follow a ARA_∞ assumption, the system will stay in a steady-state regime.
- In the static policy, the optimal duration δ^* is computed on an infinite horizon. This implies that the initial condition of the system has no impact on δ^*. The Assumption (i) is consequently not necessary and the system can be considered as good as new at the beginning of the observation.
- In the case of perfect maintenances, the planned PM strategy developed in the Assumption (iii) is called age-based preventive maintenance policy. It implies that a PM is carried out when

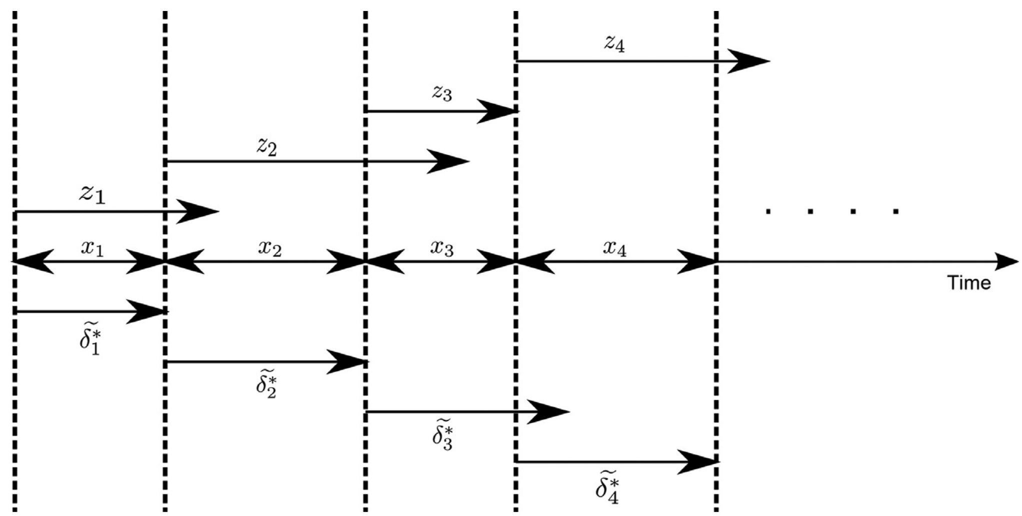


Fig. 4. Trajectory of the maintenance process for the dynamic strategy.

the virtual age of the system reaches a predetermined age, if no failure has occurred before. As maintenances are imperfect, the PM policy described in the Assumption (iii) is not based on the virtual age of the system, but on a duration since the last maintenance. The terminology of age-based PM can still be preserved (El-Ferik & Ben-Daya, 2008).

- The Assumption (vi) is conservative, as PM are generally more efficient than CM.
- The Assumption (vii) is rather strong, as the wear-out of the system and the maintenance efficiencies are not precisely known in practice.
- As commonly postulated in the Assumption (viii), the cost of a maintenance is independent of its efficiency. In practice, it is possible that the corrective maintenance cost depend of the restoration factor ρ (Yevkin & Krivtsov, 2013).

3.2.1. Policy 1: static policy

For the first strategy, the age-based PM is set to a constant value δ^*. No analytical results allow to obtain the optimal duration δ^*. Monte Carlo simulations have been used to obtain the optimum. As the resulting cost functions are relatively smooth, a grid optimization has been developed. A reasonable bound for δ^*, denoted δmax, has been set to the 95th percentile of the inter-failure distribution X_∞ developed in Proposition 8. The simulated trajectories run over a sufficiently long period $Tmax$ so that the long-run average cost per unit of time is convergent. As mentioned in Section 3.2, the system can be considered as good as new at the start. The corresponding Monte Carlo simulations have been derived and commented in Algorithm 1 in Appendix H.

A system under no PM follows the $WARA_\infty$ model. In particular, the inter-failure times converge to a limiting distribution X_∞ derived in Proposition 8. The resulting repair process can therefore be roughly approximated by a renewal process with X_∞ as the generic distribution of the inter-arrival times. The optimal age-based PM in the corresponding renewal process $\widehat{\delta^*}$ can be obtained by minimizing the mean cost per unit time as in (12) (Gertsbakh, 2000).

$$\widehat{\delta^*} = \arg\min_{\delta>0} \frac{C_p + (C_c - C_p)(1 - R_{X_\infty}(\delta))}{\int_0^\delta R_{X_\infty}(u)du} \tag{12}$$

Finkelstein (2015) shows that under the Static policy, the optimal value $\widehat{\delta^*}$ of the $WARA_\infty$ model exists. As for the optimal age-based PM in a Weibull renewal process (Tadikamalla, 1980), there is however no analytical results for $\widehat{\delta^*}$ and the mean cost per unit time function is minimized numerically.

3.2.2. Policy 2: dynamic policy

After a maintenance, the time to the next PM is scheduled according to the wear-out of the system and the past of the maintenance process. At the beginning of the observation, the system is under the stationary regime of the $WARA_\infty$ model. The effective age of the system is $\mathcal{A}$ with distribution A_∞, presented in Proposition 8. As discussed in Section 2.2, this assumption is reasonable if the system has failed at least a few times in its history. The optimal PM duration $\widetilde{\delta_1^*}$ for the first cycle is derived by minimizing (12). Given the initial age $\mathcal{A} = u$, the effective age of the system A_i after the *ith* maintenance (PM or CM) can be derived as in (13).

$$A_i(u) = (1-\rho)^i u + \sum_{j=1}^{i} (1-\rho)^{i-j+1} x_j \tag{13}$$

Let us denote by Z and Z_{i+1} the potential time to failure for a new system and the potential inter-failure time consecutive to the *ith* maintenance, respectively. Given the history of the maintenance process, the conditional distribution of Z_{i+1} can be derived as in (14) by conditioning on $\mathcal{A}$. Note that the initial age $\mathcal{A}$ is assumed to be independent of the future of the process and that the effective age A_i has been obtained in (13).

$$\begin{aligned} P(Z_{i+1}>z|X_1,\ldots,X_i) &= \int_0^\infty P(Z_{i+1} > z|X_1,\ldots,X_i,\mathcal{A}=u) - dR_{A_\infty}(u) \\ &= \int_0^\infty P(Z > A_i(u)+z|Z > A_i(u)) - dR_{A_\infty}(u) \\ &= -\int_0^\infty \frac{e^{-\alpha(A_i(u)+z)^\beta}}{e^{-\alpha A_i(u)^\beta}} dR_{A_\infty}(u) \end{aligned} \tag{14}$$

As the conditional distribution of the potential next inter-failure time is known, a preventive maintenance policy can be adjusted accordingly. The optimization of the implementation of the next PM is similar to the case of a renewal process discussed at the end of Section 3.2.1. The optimal PM duration $\widetilde{\delta_{i+1}^*}$ for the $(i+1)$th cycle has been derived in (15). As for $\widehat{\delta^*}$ in Section 3.2.1, $\widetilde{\delta_{i+1}^*}$ is obtained numerically.

$$\widetilde{\delta_{i+1}^*} = \arg\min_{\delta>0} \frac{C_p + (C_c - C_p)(1 - P(Z_{i+1} > \delta|X_1,\ldots,X_i))}{\int_0^\delta P(Z_{i+1} > u|X_1,\ldots,X_i)du} \tag{15}$$

The actual inter-maintenance time x_{i+1} is associated with the event which occurs first $x_{i+1} = min(\widetilde{\delta_{i+1}^*}, z_{i+1})$. An example of trajectory of the maintenance process is presented in Fig. 4. The average cost C^{DYN} of the dynamic policy on an infinite horizon can be

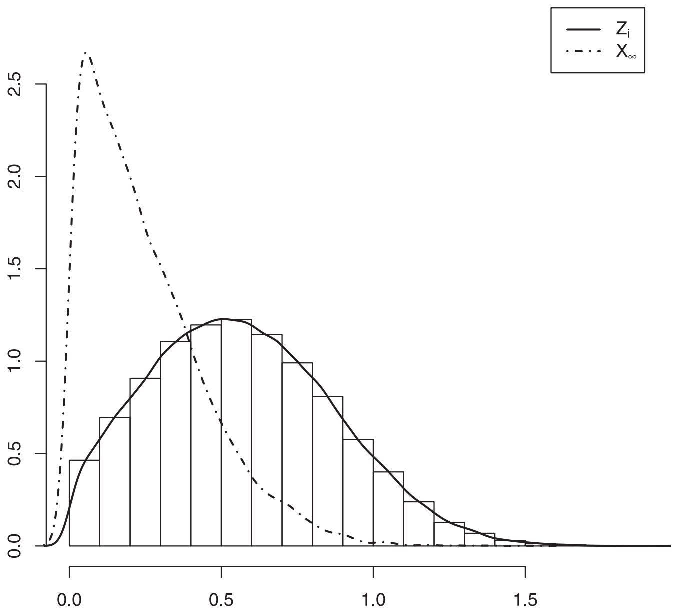


Fig. 5. Density of inter-maintenance durations of static policy $\alpha = 1, \beta = 3, \rho = 0.2$.

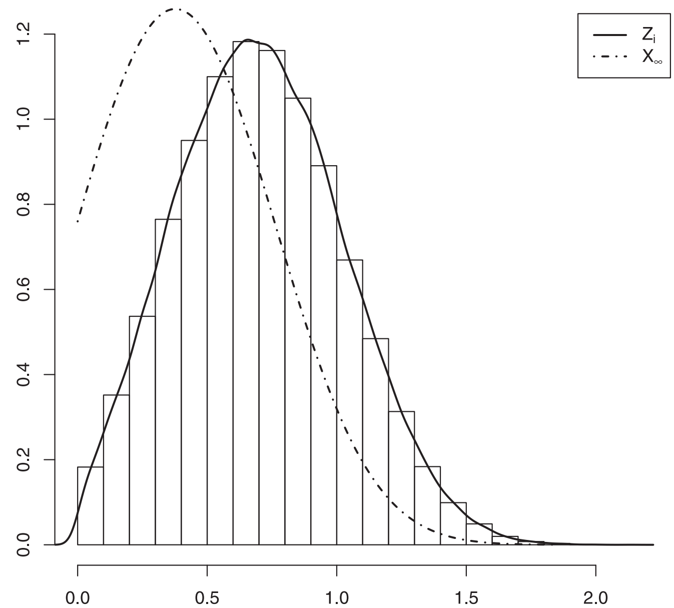


Fig. 6. Density of inter-maintenance durations of static policy $\alpha = 1, \beta = 3, \rho = 0.5$.

assessed from its empirical version as in (16).

$$C^{DYN} = \frac{C_c \sum_{i\geq 1} \mathbb{1}_{Z_i<\widetilde{\delta}_i^*} + C_p \sum_{i\geq 1} \mathbb{1}_{Z_i>\widetilde{\delta}_i^*}}{\sum_{i\geq 1} \mathbb{1}_{Z_i<\widetilde{\delta}_i^*} Z_i + \sum_{i\geq 1} \mathbb{1}_{Z_i>\widetilde{\delta}_i^*} \widetilde{\delta}_i^*} \tag{16}$$

3.2.3. Policy 3: failure limit policy

The policy uses the idea of a failure limit policy, in which a PM is performed when the failure rate reaches a maximum level λ^* (Jayabalan & Chaudhuri, 1992; Lie & Chun, 1986; Malik, 1979). This allows to keep the system's failure rate below an acceptable level. To apply to the virtual age models, we first observe that the failure rate at time t can be determined by the corresponding virtual age: $\lambda_t = \lambda(V_t)$. It follows that a maximum (might be optimal) level of failure rate λ^* corresponds to a maximum level of virtual age $V_t^* = s^*$. Considering the virtual age assumption, the failure limit policy performs a PM when the virtual age of the system exceed a maximum threshold s^*. The PM duration of the $(i+1)th$ cycle is determined by $\delta_{i+1} = s^* - a_i$, if there is no failure. We propose an algorithm using Monte-Carlo simulations in Appendix I to determine the optimal value s^* of the virtual age. The optimal threshold aims to minimize the long-term average cost per unit of time. The time limit $Tmax$ is sufficiently large to ensure the convergence.

3.2.3.1. ***Remark****.* Let us stress that in the maintenance policies under consideration, there is a competing risks involved. For example, in the static policy, if the ith time to failure is Z_i, the current time to maintenance is $X_i = min(Z_i, \delta)$. As the preventive maintenances are planned, this competing relationship can be modeled as a deterministic censoring of the time to failure. The results from Section 2 derive the properties in the steady-state regime under one kind of maintenance. Considering two kinds of maintenances and as the preventive maintenance are a constant and deterministic censoring of the failure process, the resulting process will naturally converge to a stationary regime. However, the distribution of the limiting virtual age A_∞ will not be the same as for one kind of maintenance. Theoretical results have been obtained for the stationary regime under failure-limit or age-dependent preventive maintenance policy, but not in the case of the static and dynamic policies from the paper. Through graphical presentation in Figs. 5–7, we illustrate the stationary assumption after applying ARA_∞ preventive maintenance actions.

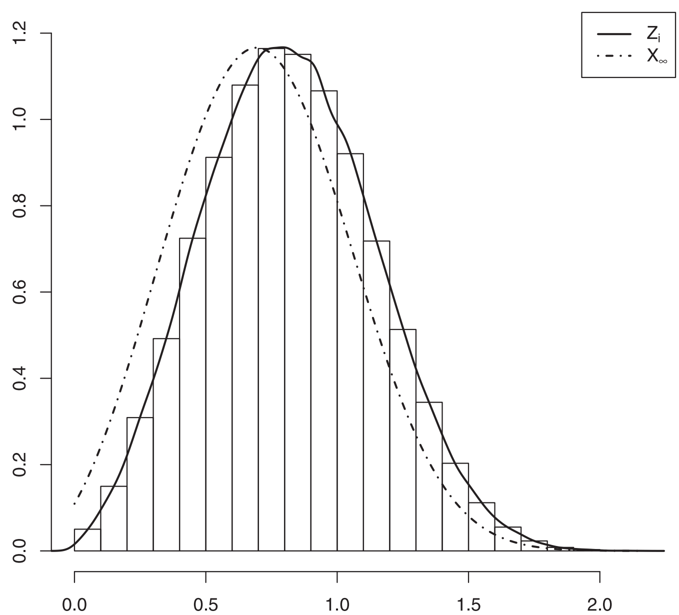


Fig. 7. Density of inter-maintenance durations of static policy $\alpha = 1, \beta = 3, \rho = 0.8$.

3.3. Numerical simulations

The implementation of the different PM strategies are discussed in this section based on representing examples. Since the failure data set of electrical transformers given by French electrical company (Électricité de France-EDF) is confidential, the proposed theoretical results are applied to simulated data with parameter setting similar to the original data set. Furthermore, simulated data with different parameters setting permit a better analyze of the maintenance policies efficiency.

Different wear rates β and different restoration factors ρ are considered. The pseudo-scale parameter α is set to 1. Nine configurations are chosen with $\beta \in \{1.5, 3, 4.5\}$ and $\rho \in \{0.2, 0.5, 0.8\}$. These choices cover slow to fast aging and poorly to fairly efficient maintenances. Three different cost ratios $C_c/C_p \in \{10, 100, 1000\}$ are considered. The Nelder–Mead downhill simplex method is used for the optimization procedure.The results for all the strategies are

Table 1
Comparing costs of maintenance policies ($\alpha = 1, C_c = 10C_p$).

β	ρ	Static		Failure limit		Variant		Dynamic	No PM	$E[X_\infty]$
		Cost	δ^*	Cost	s^*	Cost	$\widehat{\delta^*}$	C^{DYN}		
1.5	0.2	17.72	0.18	17.69	0.85	20.81	0.84	20.76	22.01	0.45
1.5	0.5	12.30	0.26	12.28	0.50	13.51	0.62	13.11	15.68	0.64
1.5	0.8	9.71	0.33	9.71	0.41	9.97	0.48	9.71	12.74	0.78
3	0.2	15.50	0.09	15.48	0.48	25.00	0.23	23.64	40.70	0.24
3	0.5	7.54	0.20	7.54	0.40	8.02	0.26	7.62	20.72	0.48
3	0.8	4.92	0.30	4.92	0.38	4.92	0.30	4.92	13.90	0.72
4.5	0.2	12.51	0.10	12.51	0.51	16.51	0.15	16.10	46.87	0.21
4.5	0.5	5.49	0.23	5.48	0.46	5.51	0.22	5.49	21.39	0.47
4.5	0.8	3.46	0.37	3.46	0.46	3.49	0.34	3.46	13.68	0.73

Table 2
Optimal maintenance strategies in nine configurations with ($\alpha = 1, C_c = 100C_p$).

β	ρ	Static		Failure limit		Variant		Dynamic	No PM
		Cost	δ^*	Cost	s^*	Cost	$\widehat{\delta^*}$	C^{DYN}	
1.5	0.2	87.46	0.03	87.46	0.17	88.92	0.05	88.65	220
1.5	0.5	60.52	0.05	60.50	0.10	62.19	0.07	62.05	156
1.5	0.8	47.61	0.06	47.52	0.08	49.68	0.10	48.21	127
3	0.2	34.45	0.04	34.41	0.22	41.61	0.06	38.65	407
3	0.5	16.75	0.09	16.73	0.17	16.92	0.08	16.80	207
3	0.8	10.91	0.13	10.90	0.17	11.10	0.11	11.09	139
4.5	0.2	21.40	0.06	21.32	0.30	23.19	0.05	22.95	468
4.5	0.5	9.33	0.13	9.33	0.27	12.74	0.08	9.35	213
4.5	0.8	5.89	0.21	5.89	0.27	6.62	0.16	5.96	136

Table 3
Optimal maintenance strategies in nine configurations with ($\alpha = 1, C_c = 1000C_p$).

β	ρ	Static		Failure limit		Variant		Dynamic policy	No PM
		Cost	δ^*	Cost	s^*	Cost	$\widehat{\delta^*}$	C^{DYN}	
1.5	0.2	408	0.007	407	0.03	501	0.03	471	2200
1.5	0.5	282	0.009	282	0.02	309	0.02	291	1568
1.5	0.8	221	0.012	221	0.01	233	0.02	223	1274
3	0.2	74.33	0.02	74.20	0.10	87.23	0.02	77.15	4070
3	0.5	36.11	0.04	36.03	0.08	41.40	0.03	38.23	2072
3	0.8	23.52	0.06	23.52	0.08	28.32	0.04	25.02	1390
4.5	0.2	35.64	0.03	35.59	0.18	66.92	0.01	40.16	4687
4.5	0.5	15.58	0.08	15.56	0.16	36.86	0.03	17.05	2139
4.5	0.8	9.85	0.13	9.85	0.16	14.17	0.07	9.91	1368

reported in Tables 1–3. The average cost when no PM is performed (only repair at failure) for an infinite horizon is added for comparison purpose. In Table 1, the expectation of the inter-failure time in stationary regime is also included. The result of the approximation by renewal theory for the static policy is referred as the Variant in the table.

It is obvious to see that any maintenance policy without planned preventive maintenance (only repair at failure) is the most costly. The PM clearly helps to lower the cost and extend the operating time. It is interesting to evaluate the impact of preventive maintenance on the average cost in different policies.

The failure limit policy is the most cost-wise policy. We know that, by definition of the policy, the PM durations are not constant. However, when obtaining numerical results, we observe that almost all of the PM durations are of the same length. The small portion of different lengths is meant to adjust the system's virtual age to its optimal value. We believe that this behavior is due to the particularity of the ARA_∞ model.

As for the static policy, the cost functions for all the configurations are derived from Algorithm 1 and are presented in Figs. 8–10. The optimal solution δ^* and its approximation from the renewal process theory (Variant) $\widehat{\delta^*}$ are indicated. For large values of β (3,4.5), the Variant proposes good approximation of the op-

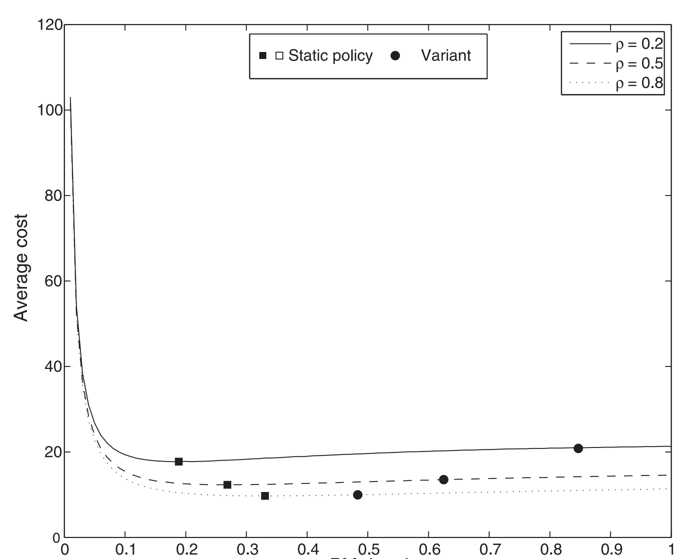


Fig. 8. Average cost of age-based policy with $\alpha = 1, \beta = 1.5$.

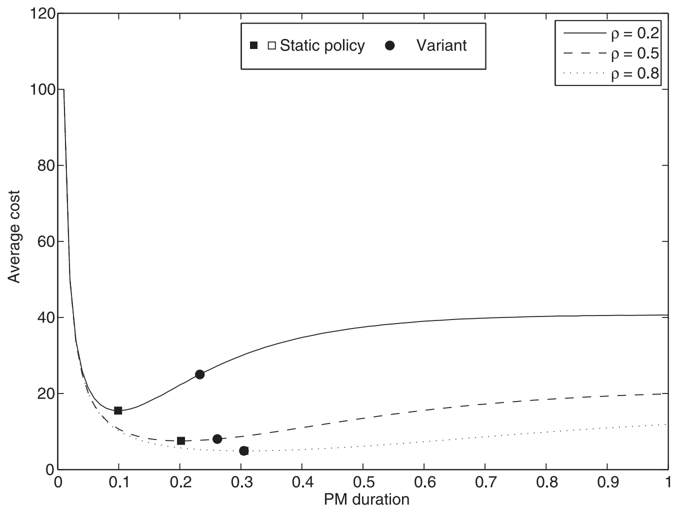


Fig. 9. Average cost of age-based policy with $\alpha = 1, \beta = 3$.

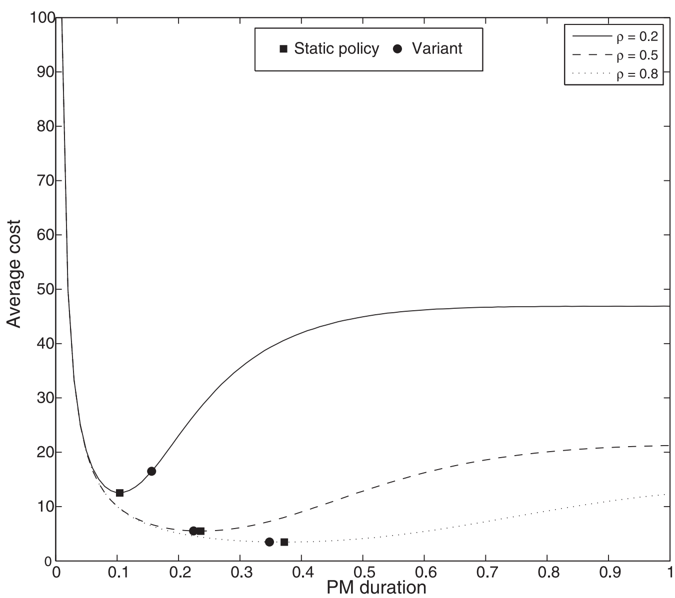


Fig. 10. Average cost of age-based policy with $\alpha = 1, \beta = 4.5$.

timal solution obtained by Monte-Carlo simulations. In particular, when repair is effective ($\rho = 0.8$), the Variant returns fairly correct approximation of the average cost and the optimal PM duration. From numerical results, we observe that the static policy, though simpler, is almost as effective as the failure limit policy.

The dynamic policy is better than the approximation from renewal theory (Variant), but is outperformed by the static policy with Monte-Carlo simulations when repair is inefficient. For large values of β and ρ, the dynamic policy seems to approach to the static policy's level. The dynamic policy is locally optimal, but is no longer optimal in the infinite horizon. An example of the empirical distribution of the PM durations under dynamic policy comparing to the static policy is given in Figs. 11–13.

In conclusion, the failure limit policy is the most cost efficient policy but it is also the most difficult to apply because one has to measure the system's virtual age, not the real age. The static policy is almost as efficient as the first policy, with one major advantage in its implementation: the PM durations are fixed and can be determined in advance. The dynamic policy is not as effective as the static one in infinite horizon, but it is locally optimal. It suggests that the dynamic policy could be of interest for systems with limited number of failures before a replacement. This criteria could be imposed, for instance, by design or by safety measure.

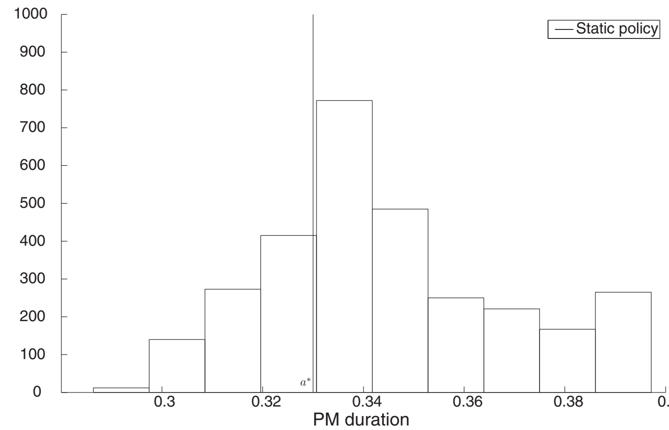


Fig. 11. Comparison of dynamic optimal PM durations and optimal PM duration of static policy with ($\beta = 1.5, \rho = 0.8$).

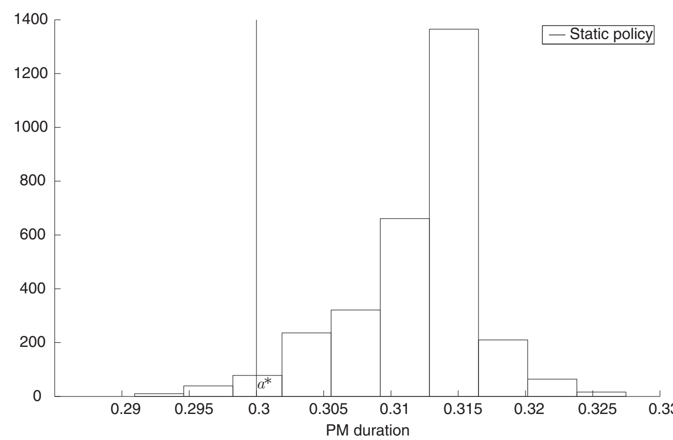


Fig. 12. Comparison of dynamic optimal PM durations and optimal PM duration of static policy with ($\beta = 3, \rho = 0.8$).

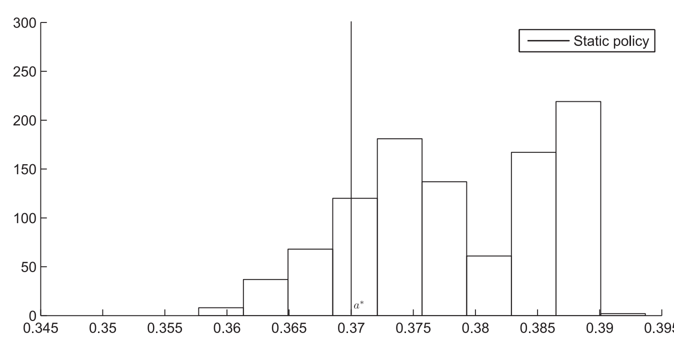


Fig. 13. Comparison of dynamic optimal PM durations and optimal PM duration of static policy with ($\beta = 4.5, \rho = 0.8$).

Let us note that, the computations in the dynamic case are much more important and it is actually necessary to constantly update the virtual age of the system. However, these computations are not implying intensive Monte Carlo simulations and can be done rapidly and extremely efficiently. Furthermore, in practice, the computational time to obtain the optimal solution is matter of milliseconds, which is a different order of magnitude from the time to planned preventive maintenance. Moreover, it has been highlighted that the static policy offers much better results than the dynamic one. The static policy does not require to compute the virtual age of the system after each maintenance and is consequently very simple to implement. The only reason to "update" the static maintenance policy would be during an on-line analysis of the system where the parameters of the models are unknown. As inference is presented as an important prospect but is not taken into account in this paper, the resulting computational issues will be the object of future research.

It can be highlighted that the parameters estimation is an important issue and is a crucial step in the course of the analysis of a maintenance data set. The inference in presence of virtual age assumptions has been well discussed in the literature (Dijoux & Idée, 2013; Doyen & Gaudoin, 2004, 2006; Lindqvist, 2006) considering one or multiple kinds of maintenance. The quality of the ML estimators for Arithmetic Reduction of Age models have been specifically discussed in Doyen (2010); Doyen and Gaudoin (2004). As our paper does not focus on inference procedures, we have not provided extensive details on how to estimate the parameters of the model but we give a brief outline of the procedure in Appendix J.

4. Conclusion

In this paper, we analyze the imperfect repair model with Weibull failure distribution under ARA_∞ assumption. We develop the marginal distributions of effective ages and inter-failure times and show the existence of a steady state. When the model reaches this state, the effective ages and the inter-failure times converge to its limiting distributions. The results are then applied to propose a static, a dynamic and a failure limit maintenance policy. Numerical simulations are presented to illustrate the policies. It is shown that the failure limit policy is the most cost effective, but it is the most difficult to implement. The static policy is almost as powerful as the first policy, with one major advantage in its application. We also observe that the dynamic policy is more effective under finite horizon planning. Heuristic algorithms are introduced to derive optimal cost and preventive maintenance durations for the policies. In all applications, the preventive maintenance and the corrective maintenance are assumed to have the same efficiency, but it is rarely the case in practice. In further research, we plan to drop this assumption in order to provide a more general framework and we believe that the convergence property of the model is still preserved. It is also interesting to investigate the ARA_∞ model with other failure distribution than the Weibull distribution. It seems that optimal maintenance policies exist for other failure distributions, although explicit distributions for reliability quantities might not be derived. We expect to determine a class of failure distributions that allows us to build optimal maintenance policies for the ARA_∞ model.

In short, the paper develops the analytical expression of the distribution of effective ages and inter-failure times during the transient and steady-state regime. In particular, these expressions confirm and detail the convergence in law of the effective ages and of the inter-failure times to a steady regime.

It is possible to derive the Residual Useful Life (RUL) of the system if the number of maintenance in the past is known or unknown and without the knowledge of the previous maintenance times. In other papers with virtual ages, the RUL is usually computed if all the maintenance history is known, which is in practice not always the case.

The results can be also applied in statistical inference framework. The maximum likelihood methods from a system observed in a time interval requires to know the initial (virtual) age of the system. Our paper allows to derive the maximum likelihood function analytically and without using burdensome Monte Carlo simulations to compute and maximize the likelihood function. The estimation of the parameters is the next natural development from the current paper (sensitivity analysis from the estimation and maintenance costs, estimation over an interval, etc) and can (will) be the subject of future works.

Moreover, the expressions of the limiting distributions allow to approximate analytically and relatively efficiently the optimal static policy. The use of virtual age is extremely convenient as the models take into account the maintenance efficiency and allow to carry out Monte Carlo simulations and numerical computations very easily. The theoretical development of particular virtual age models is important for future relevant topics in imperfect maintenance analysis. The notion of confidence interval is poorly addressed considering virtual age models. Also, there is no statistical test for virtual age model selection. In future works we can invertigate more analytical results on the behavior of imperfect maintenance model in order to bring new contributions in statistical tests and confidence intervals.

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Appendix A. Proof of Proposition 1.

The conditional distribution of X_{i+1} in (3) can be reformulated as in (A.1).

$$\forall i \geq 0, \forall t \geq 0,\ P(X_{i+1} > t | X_1, \ldots, X_i) = e^{-\Lambda(A_i+t)+\Lambda(A_i)} \tag{A.1}$$

Let us denote U_{i+1} a random variable uniformly distributed between 0 and 1 and independent of the past of the repair process. From the inverse transformation method, a simulation of X_{i+1} conditionally to the past can be obtained from (A.1) as in (A.2).

$$X_{i+1} = \Lambda^{-1}(\Lambda(A_i) - ln(U_{i+1})) - A_i \tag{A.2}$$

The random variable ξ_{i+1} defined by $\xi_{i+1} = -ln(U_{i+1})$ has an exponential distribution with rate 1, independent of the past of the repair process. A simulation of A_{i+1} can then be derived from (6) as in (A.3).

$$\begin{aligned} A_{i+1} &= (1-\rho)(A_i + X_{i+1}) \\ &= (1-\rho)(A_i + \Lambda^{-1}(\Lambda(A_i) + \xi_{i+1})) - A_i) \\ &= (1-\rho)\Lambda^{-1}(\Lambda(A_i) + \xi_{i+1}) \end{aligned} \tag{A.3}$$

This concludes the proof of Proposition 1. □

Appendix B. Proof of Proposition 2.

The initial effective age of the system A_0 is zero. The base case for A_1 is obtained by applying Proposition 1 as in (B.1).

$$A_1 = (1-\rho)\Lambda^{-1}[\Lambda(A_0) + \xi_1] = (1-\rho)\alpha^{-\frac{1}{\beta}}\xi_1^{\frac{1}{\beta}} \tag{B.1}$$

Let us assume that the proposition holds for $n \geq 1$. The induction step can be derived as in (B.2).

$$A_{n+1} = (1-\rho)\Lambda^{-1}[\Lambda(A_n) + \xi_{n+1}]$$

$$= (1-\rho)\alpha^{-\frac{1}{\beta}}\left[\alpha\left[(1-\rho)\alpha^{-\frac{1}{\beta}}\left(\sum_{i=1}^{n}(1-\rho)^{\beta(n-i)}\xi_i\right)^{\frac{1}{\beta}}\right]^{\beta} + \xi_{n+1}\right]^{\frac{1}{\beta}}$$

$$= (1-\rho)\alpha^{-\frac{1}{\beta}}\left[\alpha\alpha^{-1}(1-\rho)^{\beta}\sum_{i=1}^{n}(1-\rho)^{\beta(n-i)}\xi_i + \xi_{n+1}\right]^{\frac{1}{\beta}}$$

$$= (1-\rho)\alpha^{-\frac{1}{\beta}}\left(\sum_{i=1}^{n+1}(1-\rho)^{\beta(n+1-i)}\xi_i\right)^{\frac{1}{\beta}} \tag{B.2}$$

This completes the proof of Proposition 2. □

Appendix C. Proof of Proposition 3.

Let us denote $q=(1-\rho)^\beta$, $\theta_i=q^{i-n}$ and $Y_n=\sum_{i=1}^n\theta_i^{-1}\xi_i$. The effective ages from Proposition 2 can be expressed as in (C.1) .

$$A_n = (1-\rho)\alpha^{-\frac{1}{\beta}}\left(\sum_{i=1}^{n}\theta_i^{-1}\xi_i\right)^{\frac{1}{\beta}} = (1-\rho)\alpha^{-\frac{1}{\beta}}Y_n^{\frac{1}{\beta}} \tag{C.1}$$

As ξ_i follows an exponential distribution with rate 1, $\theta_i^{-1}\xi_i$ follows an exponential distribution with rate θ_i. A sum of n independent exponential random variables follows a hypo-exponential distribution. The distribution of Y_n follows therefore a hypo-exponential distribution and its reliability can be obtained from Gaudoin and Ledoux (2007) and is expressed in (C.2).

$$R_{Y_n}(t)=\sum_{i=1}^n\frac{\mu_i}{\theta_i}e^{-\theta_i t} \tag{C.2}$$

where $\mu_i=\frac{\prod_{i=1}^n\theta_i}{\prod_{j=1,j\neq i}^n(\theta_j-\theta_i)}$.

Using the q-shifted factorial defined in (9), a synthetic expression of μ_i can be derived as in (C.3).

$$\begin{aligned}\mu_i &= \frac{\prod_{i=1}^n\theta_i}{\prod_{j=1,j\neq i}^n(\theta_j-\theta_i)} = \frac{\prod_{i=1}^n\theta_i}{\prod_{j=1,j\neq i}^n\theta_j\left(1-\frac{\theta_i}{\theta_j}\right)} = \frac{\theta_i}{\prod_{j=1,j\neq i}^n\left(1-\frac{\theta_i}{\theta_j}\right)}\\
&= \frac{\theta_i}{\prod_{j=1}^{i-1}\left(1-\frac{\theta_i}{\theta_j}\right)\prod_{j=i+1}^n\left(1-\frac{\theta_i}{\theta_j}\right)}\\
&= \frac{q^{i-n}}{\prod_{j=1}^{i-1}(1-q^{i-j})\prod_{j=i+1}^n(1-q^{i-j})}\\
&= \frac{1}{q^{n-i}}\frac{1}{\prod_{j=1}^{i-1}(1-q^j)}\frac{1}{\prod_{j=1}^{n-i}\left(1-\frac{1}{q^j}\right)}\\
&= \frac{1}{q^{n-i}}\frac{1}{(q,q)_{i-1}}\frac{1}{\left(\frac{1}{q},\frac{1}{q}\right)_{n-i}}\end{aligned} \tag{C.3}$$

The survival function of Y_n can be expressed as in (C.4). Based on (C.1), A_n is a basic transformation of the random variable Y_n and its expression is derived in (C.5).

$$R_{Y_n}(t)=\sum_{i=1}^n\frac{1}{(q,q)_{i-1}\left(\frac{1}{q},\frac{1}{q}\right)_{n-i}}e^{-\frac{t}{q^{n-i}}} \tag{C.4}$$

$$R_{A_n}(t)=\sum_{k=1}^n\frac{1}{(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}e^{-\frac{\alpha t^\beta}{q^k}} \tag{C.5}$$

This completes the proof of Proposition 3. □

Appendix D. Proof of Proposition 4.

Given A_n, the distribution of X_{n+1} can be easily derived from (3). The reliability of X_{n+1} is consequently obtained by conditioning with respect to A_n as in (D.1).

$$\begin{aligned}P(X_{n+1}>t) &= \int_0^\infty P(X_{n+1}>t|A_n=x)dF_{A_n}(x)\\
&= \int_0^\infty P(Z>x+t|Z>x)dF_{A_n}(x)\\
&= \int_0^\infty\frac{R(x+t)}{R(x)}dF_{A_n}(x)\\
&= \int_0^\infty e^{-\alpha(x+t)^\beta+\alpha x^\beta}\alpha\beta x^{\beta-1}\\
&\quad\times\sum_{k=1}^n\frac{1}{q^k(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}e^{-\frac{\alpha x^\beta}{q^k}}dx\\
&= \sum_{k=1}^n\frac{1}{q^k(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}\\
&\quad\times\int_0^\infty\alpha\beta x^{\beta-1}e^{-\alpha(x+t)^\beta+\alpha(1-q^{-k})x^\beta}dx\end{aligned} \tag{D.1}$$

This completes the proof of Proposition 4. □

Appendix E. Proof of Proposition 5.

The Gamma function $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$ can be used to compute integrals such as in (E.1).

$$\int_0^\infty e^{-\frac{\alpha t^\beta}{q^k}}dt=\alpha^{-\frac{1}{\beta}}\Gamma\left(\frac{1}{\beta}+1\right)q^{\frac{k}{\beta}} \tag{E.1}$$

The expectation of A_n can be obtained by combining results from Proposition (3) and (E.1). Its expression is developed in (E.2).

$$\begin{aligned}E[A_n] &= \int_0^\infty R_{A_n}(t)dt=\int_0^\infty\sum_{k=1}^n\frac{1}{(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}e^{-\frac{\alpha t^\beta}{q^k}}dt\\
&= \sum_{k=1}^n\frac{1}{(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}\int_0^\infty e^{-\frac{\alpha t^\beta}{q^k}}dt\\
&= \alpha^{-1/\beta}\Gamma\left(\frac{1}{\beta}+1\right)\sum_{k=1}^n\frac{q^{\frac{k}{\beta}}}{(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}\end{aligned} \tag{E.2}$$

The expectation of X_{n+1} can be obtained by integrating the associated reliability function developed in Proposition 4. A different approach is considered, based on the relationship between X_{n+1} and the virtual ages A_n and A_{n+1}. An expression of $E[X_{n+1}]$ can be derived from (6) as in (E.3).

$$\begin{aligned}E[X_{n+1}] &= q^{\frac{-1}{\beta}}E[A_{n+1}]-E[A_n]\\
&= \alpha^{-1/\beta}\Gamma\left(\frac{1}{\beta}+1\right)\\
&\quad\times\left(\sum_{k=1}^{n+1}\frac{q^{\frac{k-1}{\beta}}}{(q,q)_{n+1-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}-\sum_{k=1}^n\frac{q^{\frac{k}{\beta}}}{(q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}\right)\\
&= \alpha^{-1/\beta}\Gamma\left(\frac{1}{\beta}+1\right)\\
&\quad\times\left(\sum_{k=1}^n\frac{q^{\frac{k-1}{\beta}}}{\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}\left[\frac{1}{(q,q)_{n+1-k}}-\frac{q^{\frac{1}{\beta}}}{(q,q)_{n-k}}\right]+\frac{q^{\frac{n}{\beta}}}{\left(\frac{1}{q},\frac{1}{q}\right)_n}\right)\\
&= \alpha^{-1/\beta}\Gamma\left(\frac{1}{\beta}+1\right)\\
&\quad\times\left(\sum_{k=1}^n\frac{q^{\frac{k-1}{\beta}}}{\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}\left[\frac{1-q^{\frac{1}{\beta}}(1-q^{n+1-k})}{(q,q)_{n+1-k}}\right]+\frac{q^{\frac{n}{\beta}}}{\left(\frac{1}{q},\frac{1}{q}\right)_n}\right)\\
&= \alpha^{-1/\beta}\Gamma\left(\frac{1}{\beta}+1\right)\\
&\quad\times\left(\sum_{k=1}^n\frac{q^{\frac{k-1}{\beta}}\left(1-q^{\frac{1}{\beta}}+q^{n+1-k+\frac{1}{\beta}}\right)}{(q,q)_{n+1-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}}+\frac{q^{\frac{n}{\beta}}}{\left(\frac{1}{q},\frac{1}{q}\right)_n}\right)\\
&= \alpha^{-1/\beta}\Gamma\left(\frac{1}{\beta}+1\right)\end{aligned}$$

$$\times \left(\sum_{k=1}^{n+1} \frac{q^{\frac{k-1}{\beta}} \left(1 - q^{\frac{1}{\beta}} + q^{n+1-k+\frac{1}{\beta}}\right)}{(q,q)_{n+1-k} \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} \right) \tag{E.3}$$

This completes the proof of Proposition 5. □

Appendix F. Proof of Proposition 7.

The distribution of an effective age A_{n+1} is characterized by the knowledge of the previous effective age A_n, independently of n. The sequence $\{A_n\}_{n \geq 1}$ is a Markov chain on the continuous state space $\mathbb{R}_+$, defined by the initial age $A_0 = 0$ and the Markov kernel as in (F.1).

$$\begin{aligned} P(A_{n+1} > s | A_n = t) &= P((1-\rho)(A_n + X_{n+1}) > s | A_n = t) \\ &= P(X_{n+1} > \frac{s}{1-\rho} - t | A_n = t) \\ &= \begin{cases} 1 & \text{if } s < (1-\rho)t \\ P(Z > \frac{s}{1-\rho} | Z \geq t) & \text{if } s \geq (1-\rho)t \end{cases} \\ &= \frac{R\left(\frac{s}{1-\rho}\right)}{R(t)} \mathbb{1}_{\{s \geq (1-\rho)t\}} + \mathbb{1}_{\{s < (1-\rho)t\}} \\ &= e^{-\alpha((\frac{s}{1-\rho})^\beta - t^\beta)} \mathbb{1}_{\{s \geq (1-\rho)t\}} + \mathbb{1}_{\{s < (1-\rho)t\}} \end{aligned} \tag{F.1}$$

The transition density $p(s, t)$ is obtained by taking the derivative with respect to s and by modifying the sign. This completes the proof of Proposition 7. □

Appendix G. Proof of Proposition 8.

Regarding the limiting distribution of the effective age, the existence of the survival function R_{A_∞} will be proven. Then the convergence in distribution of the effective ages to A_∞ will be obtained. Basic considerations are first introduced and will be helpful for the proof.

(I) The particular q-shifted factorial $(q, q)_k$ converges in the interval $(0, 1)$ to the Euler function $(q, q)_\infty$.
(II) $|(\frac{1}{q}, \frac{1}{q})_{k-1}|$ is increasing in k and tends to infinity when q is in the interval $(0, 1)$.
(III) $\forall x \geq 0,\ xe^{-x} \leq 1$.
(IV) $\lim_{n\to\infty} \sum_{k=1}^{n} \left[\prod_{j=n-k+1}^{\infty} (1 - q^j) - 1\right] q^k = 0$

The convergence of the partial sum $R_{A_n}(t)$ to the infinite sum $R_{A_\infty}(t)$ is proven in two steps: the first step consists of proving the infinite series $R_{A_\infty}(t)$ is convergent, and the second step consists of proving the convergence of $R_{A_n}(t)$ to $R_{A_\infty}(t)$.

We prove the convergence of the infinite series $R_{A_\infty}(t)$. As $(q, q)_\infty$ is positive and bounded and $|(\frac{1}{q}, \frac{1}{q})_{k-1}|$ increases towards infinity, $|\frac{1}{(q,q)_\infty (\frac{1}{q}, \frac{1}{q})_{k-1}}|$ has an upper bound M. The absolute value of the general term of the series R_{A_∞} can be bounded as in (G.1) using (III). The upper and lower bounds are general terms of convergent series, which proves the existence of R_{A_∞} for t positive. The function is also naturally defined in 0.

$$0 \leq \frac{1}{(q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} e^{-\frac{\alpha t^\beta}{q^k}} \leq M \frac{q^k}{\alpha t^\beta} \tag{G.1}$$

As for the convergence in distribution, the difference between R_{A_n} and R_{A_∞} are expressed in (G.2).

$$\begin{aligned} &|R_{A_n}(t) - R_{A_\infty}(t)| \\ &= \left| \sum_{k=1}^{n} \frac{e^{-\frac{\alpha t^\beta}{q^k}}}{(q,q)_{n-k} \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} - \sum_{k=1}^{\infty} \frac{e^{-\frac{\alpha t^\beta}{q^k}}}{(q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} \right| \\ &= \left| \sum_{k=1}^{n} \left(\frac{1}{(q,q)_{n-k}} - \frac{1}{(q,q)_\infty} \right) \frac{e^{-\frac{\alpha t^\beta}{q^k}}}{\left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} - \sum_{k=n+1}^{\infty} \frac{e^{-\frac{\alpha t^\beta}{q^k}}}{(q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} \right| \\ &= \left| \sum_{k=1}^{n} \frac{\left[\prod_{j=n-k+1}^{\infty} (1-q^j) - 1\right] e^{-\frac{\alpha t^\beta}{q^k}}}{(q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} - \sum_{k=n+1}^{\infty} \frac{e^{-\frac{\alpha t^\beta}{q^k}}}{(q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} \right| \\ &\leq \frac{M}{\alpha t^\beta} \left| \sum_{k=1}^{n} \left[\prod_{j=n-k+1}^{\infty} (1-q^j) - 1 \right] q^k \right| + \left| \sum_{k=n+1}^{\infty} \frac{e^{-\frac{\alpha t^\beta}{q^k}}}{(q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} \right| \end{aligned} \tag{G.2}$$

The first sum converges to zero from (IV). The second sum is the remainder of a convergent series and also converges to zero. Therefore, $\forall t > 0, \lim_{n\to\infty} R_{A_n}(t) - R_{A_\infty}(t) = 0$. The case $t = 0$ is trivial. The effective ages A_n are consequently convergent in distribution to A_∞.

We apply the same procedure to prove the convergence of the partial sum $R_{X_n}(t)$ to the infinite and convergent sum $R_{X_\infty}(t)$.

In the first step, the convergence of the infinite series $R_{X_\infty}(t)$ is obtained by proving that it is absolutely convergent. Considering the sum of the absolute summand:

$$\sum_{k=1}^{\infty} \left| \frac{\alpha\beta}{q^k (q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} I_k \right|$$

where

$$I_k = \int_0^\infty x^{\beta-1} e^{-\alpha(x+t)^\beta + \alpha(1-q^{-k})x^\beta} dx,\ \ k \geq 1$$

Since $t > 0$, the integral I_k is bounded by:

$$\begin{aligned} I_k &< \int_0^\infty x^{\beta-1} e^{-\alpha x^\beta + \alpha(1-q^{-k})x^\beta} dx = \int_0^\infty x^{\beta-1} e^{-\alpha q^{-k} x^\beta} dx \\ &= -\frac{1}{\alpha q^{-k}} e^{-\alpha q^{-k} x^\beta} \Big|_0^\infty = \frac{1}{\alpha q^{-k}} \end{aligned}$$

Applying the inequality to the previous series gives:

$$\begin{aligned} |R_{X_\infty}(t)| &\leq \sum_{k=1}^{\infty} \left| \frac{\alpha\beta}{q^k (q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} I_k \right| \\ &\leq \sum_{k=1}^{\infty} \frac{1}{(q,q)_\infty \left| \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1} \right|} \end{aligned}$$

As the term $|(\frac{1}{q}, \frac{1}{q})_{k-1}|$ tends to infinite after the condition (II), there exists $k_0 < \infty$ such that $\forall k \geq k_0, |1 - \frac{1}{q^k}| \geq 2$. This gives:

$$\begin{aligned} &\sum_{k=1}^{\infty} \left| \frac{\alpha\beta}{q^k (q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} I_k \right| \\ &\leq \sum_{k=1}^{k_0} \frac{1}{(q,q)_\infty \left| \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1} \right|} + \frac{1}{(q,q)_\infty} \sum_{k=k_0+1}^{\infty} \left| \frac{1}{\left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} \right| \\ &= \sum_{k=1}^{k_0} \frac{1}{(q,q)_\infty \left| \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1} \right|} \\ &\quad + \frac{1}{(q,q)_\infty} \left| \frac{1}{\left(\frac{1}{q}, \frac{1}{q}\right)_{k_0}} \right| \sum_{k=k_0}^{\infty} \left| \frac{1}{\prod_{i=k_0}^{k} \left(1 - \frac{1}{q^i}\right)} \right| \end{aligned}$$

$$\leq \sum_{k=1}^{k_0} \frac{1}{(q,q)_\infty \left|\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}\right|} + \frac{1}{(q,q)_\infty}\left|\frac{1}{\left(\frac{1}{q},\frac{1}{q}\right)_{k_0}}\right| \sum_{i=1}^{\infty} \frac{1}{2^i}$$

Since the second term of the right hand side is a convergent series, the infinite sum of the absolute summand is convergent, hence the infinite series $R_{X_\infty}(t)$ is convergent absolutely. Il implies that the series $R_{X_\infty}(t)$ is convergent.

To show the convergence of the partial sum $R_{X_n}(t)$ to the infinite sum $R_{X_\infty}(t)$, we evaluate the following term:

$$\begin{aligned}
|R_{X_n}(t) - R_{X_\infty}(t)| &= \left| \sum_{k=1}^{n} \frac{\alpha\beta}{q^k (q,q)_{n-k}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}} I_k \right.\\
&\quad \left. - \sum_{k=1}^{\infty} \frac{\alpha\beta}{q^k (q,q)_{\infty}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}} I_k \right|\\
&= \left| \sum_{k=1}^{n} \frac{\alpha\beta}{q^k \left(\frac{1}{q},\frac{1}{q}\right)_{k-1}} \left(\frac{1}{(q,q)_{n-k}} - \frac{1}{(q,q)_\infty} \right) I_k \right.\\
&\quad \left. - \sum_{k=n+1}^{\infty} \frac{\alpha\beta}{q^k (q,q)_{\infty}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}} I_k \right|\\
&\leq \left| \sum_{k=1}^{n} \frac{\alpha\beta}{q^k \left(\frac{1}{q},\frac{1}{q}\right)_{k-1}} \left(\frac{1}{(q,q)_{n-k}} - \frac{1}{(q,q)_\infty} \right) I_k \right|\\
&\quad + \left| \sum_{k=n+1}^{\infty} \frac{\alpha\beta}{q^k (q,q)_{\infty}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}} I_k \right|\\
&\leq \left| \sum_{k=1}^{n} \frac{1}{\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}} \left(\frac{1}{(q,q)_{n-k}} - \frac{1}{(q,q)_\infty} \right) \right|\\
&\quad + \left| \sum_{k=n+1}^{\infty} \frac{1}{(q,q)_{\infty}\left(\frac{1}{q},\frac{1}{q}\right)_{k-1}} \right|
\end{aligned}$$

The first sum converge to zero after the condition (IV). The second sum is the rest of the a convergent series. Consequently, the partial sum $R_{X_n}(t)$ converges to the infinite sum $R_{X_\infty}(t)$. The case where $t = 0$ is trivial.

As in Proposition 5, the expectations of A_∞ and X_∞ are obtained by integrating their respective survival function. This completes the proof of Proposition 8.

Appendix H. Algorithm for the static policy

Algorithm 1 Static policy: optimization by Monte Carlo simulations.

```
for δ ∈ [ε, δmax] with step ε do          ▷ ε > 0, precision of the algorithm
Cost(δ) = 0;   age = 0;   time = 0;
    while (time < Tmax) do
ξ = −log(RAND)                             ▷ Exponential distribution, rate 1
z = Λ^{-1}(Λ(age) + ξ) − age               ▷ Potential time to next failure
u = 𝟙_{z<δ}                                ▷ Maintenance type
Cost(δ) = Cost(δ) + C_p + (C_c − C_p)u     ▷ Update of the total cost
time = time + min(δ, z)                    ▷ Update of the current time
age = (1 − ρ)(age + min(δ, z))             ▷ Update of the effective age
    end while
Cost(δ) = Cost(δ)/time                     ▷ Long-run average cost
end for
δ* = arg min_{δ∈[ε,δmax]}(Cost(δ))
```

Appendix I. Algorithm for the failure limit policy

Algorithm 2 Failure limit policy: optimization by Monte Carlo simulations.

```
for s ∈ [ε, Smax] with step ε do           ▷ ε > 0, precision of the algorithm
Cost(s) = 0;   age = 0;   time = 0;
    while (time < Tmax) do
ξ = −log(RAND)                             ▷ Exponential distribution, rate 1
z = Λ^{-1}(Λ(age) + ξ) − age               ▷ Potential time to next failure
u = 𝟙_{age+z<s}                            ▷ Maintenance type
Cost(s) = Cost(s) + C_p + (C_c − C_p)u     ▷ Update of the total cost
time = time + min(z, s − age)              ▷ Update of the current time
age = (1 − ρ) × min(age + z, s)            ▷ Update of the effective age
    end while
Cost(s) = Cost(s)/time                     ▷ Long-run average cost
end for
s* = arg min_{s∈[ε,Smax]}(Cost(s))
```

Appendix J. Estimation procedure

We provide here main steps for estimating parameters of the $WARA_\infty$ model by the maximum likelihood method. Suppose that the model is observed in the interval $[0, t]$ with n failure times $t_1 < t_2 < \ldots < t_n$, the associated likelihood function is given as follows:

$$\mathcal{L}(\alpha,\beta,\rho;t_1,t_2,\ldots,t_n) = \prod_{i=1}^{n} \lambda(a_{i-1} + t_i - t_{i-1}) \times e^{-\sum_{i=1}^{n+1} \Lambda(a_{i-1}+t_i-t_{i-1}) - \Lambda(a_{i-1})}$$

with the convention $t_{n+1} = t$. Replacing $\lambda(\cdot)$ and $\Lambda(\cdot)$ by their expression and passing to logarithm yields:

$$\begin{aligned}
l(\alpha,\beta,\rho;t_1,t_2,\ldots t_n) &= \ln(\mathcal{L}(\alpha,\beta,\rho;t_1,t_2,\ldots,t_n))\\
&= n\ln(\alpha\beta) + (\beta-1)\sum_{i=1}^{n} \ln(a_{i-1}+t_i-t_{i-1})\\
&\quad -\alpha \sum_{i=1}^{n+1} (a_{i-1}+t_i-t_{i-1})^\beta - a_{i-1}^\beta
\end{aligned}$$

We note that each a_i is a function of ρ as in (7):

$$a_i = \sum_{j=1}^{i} (1-\rho)^{i-j+1}(t_j - t_{j-1}),\ i \geq 1$$

One of the advantages of the $WARA_\infty$ model is that it allows us to model a system in its steady state regime. Suppose that the $WARA_\infty$ is observed in this state, i.e. the observation window is of the form $[s, s+t]$. The n failure times $\tau_1 < \tau_2 < \cdots < \tau_n$ in the following likelihood function are the times elapsed from the starting point s to the failure instants. The associated likelihood function of the $WARA_\infty$ model under this configuration is given as follows (Nguyen, Dijoux, & Fouladirad, Pau, France, June 2014):

$$\begin{aligned}
&\mathcal{L}^\infty(\alpha,\beta,\rho;\tau_1,\tau_2,\ldots,\tau_n)\\
&\quad = \int_{(1-\rho)\tau_1}^{\infty} \prod_{i=1}^{n} \lambda(a_{i-1}+\tau_i-\tau_{i-1}) *\\
&\quad \times e^{-\sum_{i=1}^{n+1}[\Lambda(a_{i-1}+\tau_i-\tau_{i-1})-\Lambda(a_{i-1})]} dF_{A_\infty}(x)
\end{aligned}$$

The virtual age a_i now becomes a function of the maintenance efficiency ρ and the initial virtual age $V_s = a_0 = x$ at the starting point s:

$$a_i = x + \sum_{j=1}^{i} (1-\rho)^{i-j+1}(t_j - t_{j-1}), \quad i \geq 1$$

Three parameters (α, β, ρ) are estimated by the maximum likelihood method. Even though explicit expression of the estimators are not available, they are can be easily obtained by general optimization methods, for instances the Nelder–Mead downhill simplex, interior point method etc.

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