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KT-invexity in optimal control problems

Valeriano Antunes de Oliveira [a], Geraldo Nunes Silva [b,*], Marko Antonio Rojas-Medar [c]

[a] Faculdade de Ciências Integradas do Pontal, Universidade Federal de Uberlândia, Av. José João Dib, 2545, CEP 38302-000, Ituiutaba, MG, Brasil

[b] Depto. de Ciências de Computação e Estatística, Universidade Estadual Paulista - UNESP, Rua Cristovão Colombo, 2265, CEP 15054-000, São José do Rio Preto, SP, Brasil

[c] Depto. de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, Casilla 447, Chillán, Chile

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ABSTRACT

We extend the notion of KT-invexity from mathematical programming to the classical optimal control problem and show that this generalized invexity property is not only a sufficient condition of optimality for KT-processes (processes that obey KT-conditions below) but also a necessary condition.

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1. Introduction

A variety of real problems can be modeled as optimal control problems, so that optimal control theory turns out to be an important tool for the solution of several day-by-day problems. Some of these problems occur, for example, in electrical power production (see [1]), economy (see [2]), ecology (see [3]), medicine (see [4]) and epidemics control (see [5]), among others.

In this work we study the following optimal control problem

$$\left\{\begin{array}{l} \text{minimize} \quad F(x,u) = \int_0^T f(t,x(t),u(t))\mathrm{d}t \\ \text{subject to} \quad x_i'(t) = h_i(t,x(t),u(t)),\ t\in[0,T],\ i=1,2,\ldots,n, \\ x_i(0)=\alpha_i,\ i=1,2,\ldots,n, \\ g_j(t,u(t))\le 0,\ t\in[0,T],\ j=1,2,\ldots,k, \end{array}\right\} \tag{OCP}$$

where $f : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, $h_i : [0,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$, $i\in I := \{1,2,\ldots,n\}$, and $g_j : [0,T]\times\mathbb{R}^m\to\mathbb{R}$, $j\in J := \{1,2,\ldots,k\}$, are continuously differentiable functions, $x : [0,T]\to\mathbb{R}^n$ are piecewise smooth (state) functions and $u : [0,T]\to\mathbb{R}^m$ are piecewise continuous (control) functions and $\alpha_i\in\mathbb{R}$, $i\in I$.

Constraints of the type $|u(t)|\le 1$, $t\in[0,T]$, which frequently appears in optimal control problems, are subsumed in the above model by setting $g(t,u(t)) = |u(t)|^2 - 1$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$.

* Corresponding author. Tel.: +55 17 3221 2209; fax: +55 17 3221 2203.
E-mail addresses: valeriano@pontal.ufu.br (V.A. de Oliveira), gsilva@ibilce.unesp.br (G.N. Silva), marko@ueubiobio.cl (M.A. Rojas-Medar).

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We denote by X the space of all piecewise smooth functions from $[0, T]$ on $\mathbb{R}^n$ with the norm

$$\|x\| = \|x\|_\infty + \|Dx\|_\infty, \tag{1}$$

where $x = Dy$ if $y(t) = y(0) + \int_0^t x(s)\mathrm{d}s$; and by $U,\ V,\ W$ the spaces of all piecewise continuous functions from $[0, T]$ into $\mathbb{R}^m,\ \mathbb{R}^n,\ \mathbb{R}^k$ with the uniform norm $\|\cdot\|_\infty$, respectively.

It is possible to replace the norm defined in (1) by the weaker norm $\|x\|_{1,1} = \|x\|_\infty + \int_0^T \|Dx(s)\|\mathrm{d}t$. In this case one considers control functions in a more general space of essentially bounded functions, and trajectories of the system of differential equation of the problem (OCP) as absolutely continuous functions instead of piecewise smooth ones. We chose the spaces X, U, V and W above to keep the exposition simple and in line with [6].

We say that $p = (x, u) \in X \times U$ is a *feasible control process* if the pair (x, u) satisfies the constraints of (OCP). We say that $\bar{p} = (\bar{x}, \bar{u})$ is an *optimal control process* if $(\bar{x}, \bar{u})$ is a solution of (OCP), that is, if $F(p) \geq F(\bar{p})$ for all feasible control processes p.

A feasible control process that satisfies the classical Kuhn–Tucker type necessary optimality condition (Theorem 1 below) is called a *KT-process*.

The purpose of this paper is to extend the notion of KT-invexity from mathematical programming to optimal control problems and show that a KT-process of an optimal control problem is optimal if and only if this problem is KT-invex. The notions of KT-processes and KT-invexity will be defined in Section 3.

2. Background and preliminary results

Following Craven in [6], the differential equation $x_i'(t) = h_i(t, x(t), u(t)),\ t \in [0, T],\ x_i(0) = \alpha_i,\ \ i \in I$, can be re-written as $Dx = H(x, u)$, where $H : X \times U \to V$ is defined by $H(x, u)(t) = (h_i(t, x(t), u(t))_{i\in I})$. Defining $G : U \to W$ by $G(u)(t) = (g_j(t, u(t))_{j\in J})$, we can re-write (OCP) as the abstract problem:

$$\begin{aligned} &\text{minimize} \quad F(x, u)\\ &\text{subject to} \quad Dx = H(x, u),\\ &G(u) \in -S, \end{aligned} \tag{2}$$

where $S = \{w \in W : w(t) \geq 0,\ t \in [0, T]\}$. Optimality conditions for the abstract problem above were obtained by Craven [6]. Actually assuming that the linear operator $-D + H_x(\bar{x}, \bar{u})$ is onto, Craven characterized optimal processes of the Kuhn–Tucker type.

Assuming that the operator $-D + H_x(\bar{x}, \bar{u})$ is onto is the same as assuming that the differential equation

$$\begin{cases} x_i'(t) + \nabla_x h_i(t, \bar{x}(t), \bar{u}(t))^{\mathrm{T}} x(t) = v_i(t), \quad t \in [0, T],\ i \in I,\\ x_i(0) = \alpha_i, \quad i \in I, \end{cases} \tag{LS}$$

has a solution $x \in X$ whichever $v = (v_1, v_2, \ldots, v_n) \in V$. When this happens we will say that the *Local Solvability* (LS) *Condition* holds at $\bar{p}$.

Theorem 1 (*Kuhn–Tucker*). *Let $\bar{p} = (\bar{x}, \bar{u}) \in X \times U$ be an optimal process. Assume that the* (LS) *Condition holds at $\bar{p}$. Then there exist piecewise smooth functions $\mu : [0, T] \to \mathbb{R}^n$ and $\lambda : [0, T] \to \mathbb{R}^k$ satisfying, for almost every $t \in [0, T]$,*

$$\nabla_x f(t, \bar{p}) + \sum_{i\in I} \mu_i(t)\nabla_x h_i(t, \bar{p}) + \mu'(t) = 0; \tag{3}$$

$$\mu_i(T) = 0, \quad i \in I; \tag{4}$$

$$\nabla_u f(t, \bar{p}) + \sum_{i\in I} \mu_i(t)\nabla_u h_i(t, \bar{p}) + \sum_{j\in J} \lambda_j(t)\nabla_u g_j(t, \bar{u}) = 0; \tag{5}$$

$$\lambda_j(t) g_j(t, \bar{u}) = 0, \quad j \in J; \tag{6}$$

$$\lambda_j(t) \geq 0, \quad j \in J. \tag{7}$$

Proof. See Craven [6]. □

When conditions (3)–(7) above are satisfied for a feasible process $\bar{p} \in X \times U$, we say that $\bar{p}$ is a *KT-process*. The functions $\lambda(t)$ and $\mu(t)$ are called Lagrange multipliers.

Theorem 1 is very close to the Pontryagin Maximum Principle (see, for instance, Alekseev et al. [7]), which is the main result of the optimal control theory. The condition (LS) is a constraint qualification for the abstract mathematical programming problem (2) which provides conditions for the multiplier related to the cost function to be one, avoiding degeneracy of the optimality conditions. Translating the necessary conditions from the abstract set up to the control theoretical setting we obtain the necessary conditions in the above theorem with the multiplier related to cost function still 1; see the Eqs. (3), (5). For more details, see Craven [6].

The Theorem 1 furnishes necessary optimality conditions for a pair (x, u) to be a solution of the control problem. But, without supplementary hypothesis, they are not sufficient. To decide whether a candidate process is optimal or not there are three approaches: (1) the verification function method via Hamilton–Jacobi–Bellman theory; (2) second order methods; and (3) the exploration of the special structure of the functions such as convexity, deformation to simpler problems, generalized convexity, etc. In the last years, linked with mathematical programming, the notion of invex functions and its generalizations have appeared in the literature (see [8,6,9,10]), which has been very fruitful with respect to obtaining sufficient optimality conditions. These notions have been used in continuous-time programming problems, see [11–15], and in certain classes of variational and control problems, see for instance, [16–18].

In [10], Martin introduced a generalization of the notion of invexity, which he denominated KT-invexity. Martin, then, showed that KT-invexity is, like invexity, a sufficient condition of optimality to the classical mathematical programming problem, in the sense that every stationary point (or KT-point) is a global minimizer of the problem. However, Martin's result goes further and tells us that KT-invexity is also a necessary condition for the validity of the property that every stationary point is a global minimizer. In summary, Martin showed that every stationary point is a global minimizer if and only if the problem is KT-invex. Then we see that the largest class of problems where such property is valid is the class of the KT-invex problems.

In the next section we obtain a similar result to Martin's one, for the Optimal Control Problem (OCP).

3. Characterization of KT-invexity for optimal control problems

As *invexity* will play a central role in this work, we start this section with its definition which was given by Hanson in [9].

Definition 1. Given a function $\phi : S \subset \mathbb{R}^n \to \mathbb{R}$, where S is an nonempty open set, we say that it is *invex* if there exists a map $\eta : S \times S \to \mathbb{R}^n$ satisfying

$$\phi(x) - \phi(y) \geq \nabla\phi(y)^{\mathrm{T}}\eta(x, y) \quad \forall\, x, y \in S.$$

It is clear that all convex functions are invex with $\eta(x, y) = x - y$. Other relations of invexity with the property of convexity and its generalizations can be found in Giorgi [19].

The first result, stated below as Theorem 2, furnishes sufficient optimality conditions via invexity. Before stating Theorem 2 we give the definition of invex problems.

Definition 2. We say that (OCP) is *invex* if for all $p = (x, u),\ \bar{p} = (\bar{x}, \bar{u}) \in X \times U$ there exist functions $\eta : [0, T] \to \mathbb{R}^n$, piecewise smooth, and $\xi : [0, T] \to \mathbb{R}^m$, piecewise continuous, such that

$$\int_0^{\mathrm{T}} [f(t, p) - f(t, \bar{p})]\mathrm{d}t \geq \int_0^{\mathrm{T}} [\nabla_x f(t, \bar{p})^{\mathrm{T}}\eta(t) + \nabla_u f(t, \bar{p})^{\mathrm{T}}\xi(t)]\mathrm{d}t; \tag{8}$$

$$\eta_i'(t) = x_i'(t) - h_i(t, p) - \bar{x}_i'(t) + h_i(t, \bar{p}) + \nabla_x h_i(t, \bar{p})^{\mathrm{T}}\eta(t) + \nabla_u h_i(t, \bar{p})^{\mathrm{T}}\xi(t) \quad \text{a.e. } t \in [0, T],\ i \in I; \tag{9}$$

$$\eta_i(0) = 0, \quad i \in I; \tag{10}$$

$$g_j(t, u) - g_j(t, \bar{u}) \geq \nabla_u g_j(t, \bar{u})^{\mathrm{T}}\xi(t) \quad \text{a.e. } t \in [0, T],\ j \in J. \tag{11}$$

Observe that if $f,\ h_i,\ -h_i,\ i \in I$, and $g_j,\ j \in J$, are convex, conditions (8) and (9) above are verified with $\eta(t) = x(t) - \bar{x}(t)$ and $\xi(t) = u(t) - \bar{u}(t)$. However, $\eta_i(0) = 0,\ i \in I$, is true only if p and $\bar{p}$ are feasible processes.

Theorem 2. *Suppose that (OCP) is invex. Then every KT-process is an optimal process.*

Proof. Let $\bar{p} = (\bar{x}, \bar{u}) \in X \times U$ be a KT-process. Therefore there exist Lagrange multipliers $\lambda(t)$ and $\mu(t)$ satisfying conditions (3)–(7) of Theorem 1. Taking into account that $\lambda_j(t) \geq 0$ a.e. $t \in [0, T],\ j \in J$, it follows from (8), (9) and (11) that,

$$\begin{aligned}
&\int_0^{\mathrm{T}} [f(t, p) - f(t, \bar{p})]\mathrm{d}t + \int_0^{\mathrm{T}} \sum_{j\in J} \lambda_j(t)[g_j(t, u) - g_j(t, \bar{u})]\mathrm{d}t + \int_0^{\mathrm{T}} \sum_{i\in I} \mu_i(t)[h_i(t, p) - x_i'(t) - h_i(t, \bar{p}) + \bar{x}_i'(t)]\mathrm{d}t \\
&\quad \geq \int_0^{\mathrm{T}} [\nabla_x f(t, \bar{p})^{\mathrm{T}}\eta(t) + \nabla_u f(t, \bar{p})^{\mathrm{T}}\xi(t)]\mathrm{d}t + \int_0^{\mathrm{T}} \sum_{j\in J} \lambda_j(t)\nabla_u g_j(t, \bar{u})^{\mathrm{T}}\xi(t)\mathrm{d}t \\
&\qquad + \int_0^{\mathrm{T}} \sum_{i\in I} \mu_i(t)[\nabla_x h_i(t, \bar{p})^{\mathrm{T}}\eta(t) + \nabla_u h_i(t, \bar{p})^{\mathrm{T}}\xi(t) - \eta_i'(t)]\mathrm{d}t
\end{aligned}$$

for all $p = (x, u) \in X \times U$. Using (6) and $h_i(t, \bar{p}) = \bar{x}_i'(t),\ t \in [0, T],\ i \in I$, and rearranging the terms in the second member, we obtain

$$\int_0^{\mathrm{T}} [f(t, p) - f(t, \bar{p}) + \sum_{j\in J} \lambda_j(t)g_j(t, u) + \sum_{i\in I} \mu_i(t)[h_i(t, p) - x_i'(t)]]\mathrm{d}t$$

$$\geq \int_0^T [\nabla_x f(t,\bar{p}) + \sum_{i\in I} \mu_i(t)\nabla_x h_i(t,\bar{p})]^T \eta(t)dt - \int_0^T \sum_{i\in I} \mu_i(t)\eta_i'(t)dt$$
$$+ \int_0^T [\nabla_u f(t,\bar{p}) + \sum_{j\in J} \lambda_j(t)\nabla_u g_j(t,\bar{u})]^T \xi(t)dt + \int_0^T \sum_{i\in I} \mu_i(t)\nabla_u h_i(t,\bar{p})^T \xi(t)dt$$

for all $p \in X \times U$. Using integration by parts, (4) and (10), we have

$$\int_0^T \sum_{i\in I} \mu_i(t)\eta_i'(t)dt = \sum_{i\in I} \mu_i(t)\eta_i(t)\Big|_0^T - \int_0^T \sum_{i\in I} \eta_i(t)\mu_i'(t)dt$$
$$= -\int_0^T \mu'(t)^T \eta(t)dt.$$

Substituting in the last inequality, we obtain

$$\int_0^T [f(t,p) - f(t,\bar{p}) + \sum_{j\in J} \lambda_j(t)g_j(t,u) + \sum_{i\in I} \mu_i(t)[h_i(t,p) - x_i'(t)]]dt$$
$$\geq \int_0^T [\nabla_x f(t,\bar{p}) + \sum_{i\in I} \mu_i(t)\nabla_x h_i(t,\bar{p}) + \mu'(t)]^T \eta(t)dt$$
$$+ \int_0^T [\nabla_u f(t,\bar{p}) + \sum_{j\in J} \lambda_j(t)\nabla_u g_j(t,\bar{p})]^T \xi(t)dt + \int_0^T \sum_{i\in I} \mu_i(t)\nabla_u h_i(t,\bar{p})^T \xi(t)dt$$

for all $p \in X \times U$. It follows from (3) and (5) that

$$\int_0^T [f(t,p) - f(t,\bar{p})]dt \geq -\int_0^T \sum_{j\in J} \lambda_j(t)g_j(t,u)dt - \int_0^T \left[\sum_{i\in I} \mu_i(t)[h_i(t,p) - x_i'(t)]\right] dt \tag{12}$$

for all $p \in X \times U$. In particular, if p is a feasible process, $g_j(t,u) \leq 0$, $j \in J$, and $h_i(t,p) = x_i'(t)$, $i \in I$, for $t \in [0,T]$. As $\lambda_j(t) \geq 0$ a.e. $t \in [0,T]$, $j \in J$, we get

$$\int_0^T [f(t,p) - f(t,\bar{p})]dt \geq 0$$

for all feasible processes p, such that $\bar{p}$ is an optimal process. □

Looking at the proof of the last theorem, we see that it is not necessary that the inequalities in (8) and (9) are valid for all $p, \bar{p} \in X \times U$. It is enough that they are valid for feasible processes of (OCP). In this case, the first four terms on the right hand side of (9) add up to zero. Further, because of (6), when multiplying the expression (11) by $\lambda_j(t)$, $j \in J$, the terms $\lambda_j(t)g_j(t,\bar{u}(t))$, $j \in J$, vanish if $\bar{p}$ is a KT-process. Also because of (6), we see that $\lambda_j(t) = 0$ for $t \notin A_j(\bar{u})$, where

$$A_j(\bar{u}) = \{t \in [0,T] : g_j(t,\bar{u}(t)) = 0\},\ j \in J,$$

so that it is not necessary (11) be valid for $t \notin A_j(\bar{u})$, $j \in J$. The term appearing in the right hand side of (12) is greater than or equal to zero and can be omitted with (12) still valid. Basing on these remarks, we introduce the notion of KT-invexity for Problem (OCP).

Definition 3. We say that (OCP) is *KT-invex* if for each pair of feasible processes $p = (x,u)$, $\bar{p} = (\bar{x},\bar{u}) \in X \times U$ there exist functions $\eta : [0,T] \to \mathbb{R}^n$, piecewise smooth, and $\xi : [0,T] \to \mathbb{R}^m$, piecewise continuous, such that

$$\int_0^T [f(t,p) - f(t,\bar{p})]dt \geq \int_0^T [\nabla_x f(t,\bar{p})^T \eta(t) + \nabla_u f(t,\bar{p})^T \xi(t)]dt; \tag{13}$$

$$\eta_i'(t) = \nabla_x h_i(t,\bar{p})^T \eta(t) + \nabla_u h_i(t,\bar{p})^T \xi(t) \quad \text{a.e. } t \in [0,T],\ i \in I; \tag{14}$$

$$\eta_i(0) = 0, \quad i \in I; \tag{15}$$

$$0 \geq \nabla_u g_j(t,\bar{u})^T \xi(t) \quad \text{a.e. } t \in A_j(\bar{u}),\ j \in J. \tag{16}$$

The definition above is a generalization, for the context of the Problem (OCP), of the concept of KT-invexity, introduced by Martin in [10] for the classical mathematical programming problem.

It is easy to see that if (OCP) is invex (according to the Definition 2), then it is KT-invex (according to Definition 3).

It should be noted that functions η and ξ in Definitions 2 and 3 depend on p and $\bar{p}$, that is, on (x,u) and $(\bar{x},\bar{u})$.

We would like to mention that when the endpoint condition $x_i(T) = \beta_i,\ i = 1, \ldots, n$, is present in the constraints of (OCP), for some $\beta_i \in \mathbb{R},\ i = 1, \ldots, n$, Definitions 2 and 3 need to be modified by adding $\eta_i(T) = 0,\ i = 1, \ldots, n$, so that Theorems 2 and 3 are valid.

Example 1. Let us consider the optimal control problem:

$$\begin{aligned}
&\text{minimize} \quad F(u) = \int_0^1 [1 - \exp(-u(t))]\mathrm{d}t \\
&\text{subject to} \quad x'(t) = -x(t) + u(t)^2,\ t \in [0, 1],\ x(0) = 1, \\
&u(t) \geq 0, \quad t \in [0, 1].
\end{aligned}$$

It is easy to see that $\bar{p}(t) = (\bar{x}(t), \bar{u}(t)) = (\exp(-t), 0),\ t \in [0, 1]$, is the optimal process. It is also easy to see that $\bar{p} = (\exp(-t), 0)$ is the only KT-process. So, in this problem, every KT-process is an optimal process.

This problem is not invex. Indeed, suppose, by contradiction, that there exist functions η and ξ satisfying

$$\int_0^1 [1 - \exp(-u(t)) - 1 + \exp(-\bar{u}(t))]\mathrm{d}t \geq \int_0^1 \exp(-\bar{u}(t))\xi(t)\mathrm{d}t, \tag{17}$$

$$\eta'(t) = x'(t) + x(t) - u(t)^2 - \bar{x}'(t) - \bar{x}(t) + \bar{u}(t)^2 - \eta(t) + 2\bar{u}(t)\xi(t) \text{ a.e. } t \in [0, 1], \tag{18}$$

$$-u(t) + \bar{u}(t) \geq -\xi(t) \quad \text{a.e. } t \in [0, 1], \tag{19}$$

for all $(x, u),\ (\bar{x}, \bar{u})$. Then, by (17) and (19) we have

$$\begin{aligned}
&F(u) - F(\bar{u}) - \int_0^1 \exp(-\bar{u}(t))(u(t) - \bar{u}(t))\mathrm{d}t \\
&\quad = \int_0^1 [1 - \exp(-u(t)) - 1 + \exp(-\bar{u}(t)) - \exp(-\bar{u}(t))(u(t) - \bar{u}(t))]\mathrm{d}t \\
&\quad \geq \int_0^1 [1 - \exp(-u(t)) - 1 + \exp(-\bar{u}(t)) - \exp(-\bar{u}(t))\xi(t)]\mathrm{d}t \geq 0
\end{aligned}$$

for all $u,\ \bar{u}$. In particular, for $u(t) = 0$ and $\bar{u}(t) = t,\ t \in [0, 1]$, we have

$$F(u) - F(\bar{u}) - \int_0^1 \exp(-\bar{u}(t))(u(t) - \bar{u}(t))\mathrm{d}t = 1 - 3\exp(-1) < 0,$$

which contradicts the above inequality.

We see, then, that invexity is not necessary for the validity of the property that every KT-process is an optimal process.

On the other hand, the problem is KT-invex with

$$\begin{aligned}
\eta(t) &= \eta(t, p(t), \bar{p}(t)) \\
&= 2\exp(-t) \int_0^t \bar{u}(\tau)[\exp(\tau) - \exp(\tau - u(\tau) + \bar{u}(\tau))]\mathrm{d}\tau, \\
\xi(t) &= \xi(t, p(t), \bar{p}(t)) = 1 - \exp(-u(t) + \bar{u}(t)).
\end{aligned}$$

The next lemma will be used in the proof of Theorem 3, which is our main result.

Lemma 1. *Let $\bar{p} = (\bar{x}, \bar{u}) \in X \times U$ be a feasible process. Assume that the (LS) Condition is satisfied at $\bar{p}$. If there does not exist $(z, w) \in X \times U$ such that*

$$\begin{cases}
\displaystyle\int_0^T [\nabla_x f(t, \bar{p})^{\mathrm{T}} z(t) + \nabla_u f(t, \bar{p})^{\mathrm{T}} w(t)]\mathrm{d}t < 0, \\
z_i'(t) = \nabla_x h_i(t, \bar{p})z(t) + \nabla_u h_i(t, \bar{p})w(t), \quad t \in [0, T],\ i \in I, \\
z_i(0) = 0, \quad i \in I, \\
\nabla_u g_j(t, \bar{u})w(t) \leq 0, \quad t \in A_j(\bar{u}),\ j \in J,
\end{cases}$$

then $\bar{p}$ is a KT-process of (OCP).

Proof. Define $\Phi : X \times U \to \mathbb{R},\ \gamma_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R},\ i \in I$, and $\psi_j : [0, T] \times \mathbb{R}^m \to \mathbb{R},\ j \in J$, respectively, by

$$\begin{aligned}
&\Phi(z, w) = \int_0^T [\nabla_x f(t, \bar{p})^{\mathrm{T}} z(t) + \nabla_u f(t, \bar{p})^{\mathrm{T}} w(t)]\mathrm{d}t, \\
&\gamma_i(t, z(t), w(t)) = \nabla_x h_i(t, \bar{p})z(t) + \nabla_u h_i(t, \bar{p})w(t), \quad i \in I, \\
&\psi_j(t, w(t)) = \nabla_u g_j(t, \bar{p})w(t),\ j \in J.
\end{aligned}$$

Let us consider the linear control problem

$$\begin{aligned}&\text{minimize} \quad \Phi(z,w)\\&\text{subject to} \quad z_i'(t)=\gamma_i(t,z(t),w(t)),\ t\in[0,T],\ i\in I,\\&z_i(0)=0, \quad i\in I,\\&\chi_j(t)\psi_j(t,w(t))\le 0, \quad t\in[0,T],\ j\in J,\end{aligned}$$

where $\chi_j:[0,T]\to\mathbb{R}$, is defined, for each $j\in J$, by

$$\chi_j(t)=\begin{cases}1, & \text{if } t\in A_j(\bar u),\\ 0, & \text{if } t\in[0,T]\setminus A_j(\bar u).\end{cases}$$

As, by hypothesis, there does not exist $(z,w)\in X\times U$ such that

$$\begin{cases}\int_0^T[\nabla_x f(t,\bar p)^T z(t)+\nabla_u f(t,\bar p)^T w(t)]dt<0,\\ z_i'(t)=\nabla_x h_i(t,\bar p)z(t)+\nabla_u h_i(t,\bar p)w(t), \quad t\in[0,T],\ i\in I,\\ z_i(0)=0, \quad i\in I,\\ \nabla_u g_j(t,\bar u)w(t)\le 0,\ t\in A_j(\bar u), \quad j\in J,\end{cases}$$

then we have $\Phi(z,w)\ge 0$ for all feasible process (z,w) of the problem above. On the other hand, $(\bar z,\bar w)=(0,0)$ is a feasible process and $\Phi(z,w)=0$. Therefore $(\bar z,\bar w)$ is an optimal process. It is easy to see that

$$\nabla_z\gamma_i(t,z,w)=\nabla_x h_i(t,\bar p), \qquad \nabla_w\gamma_i(t,z,w)=\nabla_u h_i(t,\bar p), \quad i\in I,$$

and

$$\nabla_w\psi_j(t,w)=\nabla_u g_j(t,\bar p), \quad j\in J.$$

Thus the problem satisfies the (LS) Condition at $(\bar z,\bar w)$. It follows from Theorem 1 that there exist piecewise smooth functions $\mu:[0,T]\to\mathbb{R}^n$ and $\lambda:[0,T]\to\mathbb{R}^k$ satisfying

$$\begin{aligned}&\nabla_x f(t,\bar p)+\sum_{i\in I}\mu_i(t)\nabla_x h_i(t,\bar p)+\mu'(t)=0,\\&\mu_i(T)=0, \quad i\in I,\\&\nabla_u f(t,\bar p)+\sum_{i\in I}\mu_i(t)\nabla_u h_i(t,\bar p)+\sum_{j\in J}\lambda_j(t)\chi_j(t)\nabla_u g_j(t,\bar u)=0,\\&\lambda_j(t)\ge 0, \quad j\in J,\end{aligned}$$

for almost every $t\in[0,T]$. Defining $\tilde\lambda_j=\chi_j\lambda_j,\ j\in J$, we have $\tilde\lambda_j(t)\ge 0$ and $\tilde\lambda_j(t)g_j(t,\bar u(t))=0$ a.e. $t\in[0,T],\ j\in J$. Hence $\bar p$ is a KT-process of (OCP). □

The next theorem is a generalization of Martin's result for the Problem (OCP).

Theorem 3. *Assume that the (LS) Condition is satisfied at each feasible process $\bar p$. Then, every KT-process is an optimal process if and only if (OCP) is KT-invex.*

Proof. Suppose that every KT-process is an optimal process. Let p and $\bar p$ be feasible processes of (OCP). If $F(p)\ge F(\bar p)$, define $\eta\equiv 0$ and $\xi\equiv 0$. Clearly conditions (13)–(16) in Definition 3 are verified. Assume that $F(p)<F(\bar p)$. Then $\bar p$ is not an optimal process and hence is not a KT-process, since we are supposing that every KT-process is optimal. It follows from Lemma 1 that there exists $(z,w)\in X\times U$ such that

$$\int_0^T[\nabla_x f(t,\bar p)^T z(t)+\nabla_u f(t,\bar p)^T w(t)]dt<0, \tag{20}$$

$$z_i'(t)=\nabla_x h_i(t,\bar p)z(t)+\nabla_u h_i(t,\bar p)w(t), \quad t\in[0,T],\ i\in I, \tag{21}$$

$$z_i(0)=0, \quad i\in I, \tag{22}$$

$$\nabla_u g_j(t,\bar u)w(t)\le 0, \quad t\in A_j(\bar u),\ j\in J. \tag{23}$$

Set

$$c=\frac{F(p)-F(\bar p)}{\int_0^T[\nabla_x f(t,\bar p)^T z(t)+\nabla_u f(t,\bar p)^T w(t)]dt}.$$

As $F(p) < F(\bar{p})$, it follows from (20) that $c > 0$. For each $t \in [0, T]$, define $\eta(t) = cz(t)$ and $\xi(t) = cw(t)$. So, by (22), $\eta_i(0) = 0,\ i \in I$, and (15) holds. We have that

$$\int_0^T [\nabla_x f(t,\bar{p})^T \eta(t) + \nabla_u f(t,\bar{p})^T \xi(t)]dt = c\int_0^T [\nabla_x f(t,\bar{p})^T z(t) + \nabla_u f(t,\bar{p})^T w(t)]dt = F(p) - F(\bar{p}).$$

Therefore

$$F(p) - F(\bar{p}) - \int_0^T [\nabla_x f(t,\bar{p})^T \eta(t) + \nabla_u f(t,\bar{p})^T \xi(t)]dt = 0,$$

so that condition (13) of Definition 3 is verified. As $c > 0$, it follows, respectively, from (21) and (23) that the conditions (14) and (16) are also verified. Thus, (OCP) is KT-invex.

Conversely, suppose that (OCP) is KT-invex. Let $\bar{p}$ be a KT-process. Therefore, there exist Lagrange multipliers $\lambda(t)$ and $\mu(t)$ satisfying conditions (3)–(7) of Theorem 1. Taking into account that $\lambda_j(t) \geq 0$ a.e. $t \in [0, T]$ and that $\lambda_j(t) = 0,\ t \in [0, T] \setminus A_j(\bar{u}),\ j \in J$, it follows from (13), (14) and (16) that,

$$\int_0^T [f(t,p) - f(t,\bar{p})]dt \geq \int_0^T [\nabla_x f(t,\bar{p})^T \eta(t) + \nabla_u f(t,\bar{p})^T \xi(t)]dt + \int_0^T \sum_{j\in J} \lambda_j(t) \nabla_u g_j(t,\bar{u})^T \xi(t)dt$$
$$+ \int_0^T \sum_{i\in I} \mu_i(t)[\nabla_x h_i(t,\bar{p})^T \eta(t) + \nabla_u h_i(t,\bar{p})^T \xi(t)]dt - \int_0^T \sum_{i\in I} \mu_i(t)\eta_i'(t)dt$$

for all feasible process p. Rearranging the terms, we obtain

$$\int_0^T [f(t,p) - f(t,\bar{p})]dt \geq \int_0^T [\nabla_x f(t,\bar{p}) + \sum_{i\in I} \mu_i(t)\nabla_x h_i(t,\bar{p})]^T \eta(t)dt$$
$$+ \int_0^T [\nabla_u f(t,\bar{p}) + \sum_{i\in I} \mu_i(t)\nabla_u h_i(t,\bar{p}) + \sum_{j\in J} \lambda_j(t)\nabla_u g_j(t,\bar{u})]^T \xi(t)dt - \int_0^T \sum_{i\in I} \mu_i(t)\eta_i'(t)dt$$

for all feasible process p. Using integration by parts, (4) and (15), we have

$$\int_0^T \sum_{i\in I} \mu_i(t)\eta_i'(t)dt = \sum_{i\in I} \mu_i(t)\eta_i(t)\Big|_0^T - \int_0^T \sum_{i\in I} \eta_i(t)\mu_i'(t)dt = -\int_0^T \mu'(t)^T \eta(t)dt.$$

Substituting in the last inequality, we obtain

$$\int_0^T [f(t,p) - f(t,\bar{p})]dt \geq \int_0^T [\nabla_x f(t,\bar{p}) + \sum_{i\in I} \mu_i(t)\nabla_x h_i(t,\bar{p})]^T \eta(t)dt + \int_0^T \mu'(t)^T \eta(t)dt$$
$$+ \int_0^T [\nabla_u f(t,\bar{p}) + \sum_{i\in I} \mu_i(t)\nabla_u h_i(t,\bar{p}) + \sum_{j\in J} \lambda_j(t)\nabla_u g_j(t,\bar{u})]^T \xi(t)dt$$

for all feasible process p. It follows from (3) and (5) that

$$\int_0^T [f(t,p) - f(t,\bar{p})]dt \geq 0$$

for all feasible process p. Hence $\bar{p}$ is an optimal process. □

4. Final considerations

In this work we showed how KT-invexity, introduced by Martin [10] in the context of mathematical programming, can be extended to optimal control problems with unilateral control constraints. We gave an example showing that the class of KT-invex control problems are larger than the class of invex control problems. The main result fully characterizes KT-invex control problems: a control problem is KT-invex iff every KT-process is an optimal process.

The assumptions made here are that all functions on the data are continuously differentiable. Relaxing this assumption to include nonsmooth functions is a topic for future research.

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