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Continuous Optimization

A generalized model for data envelopment analysis

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Abstract

Data envelopment analysis (DEA) is a method to estimate a relative efficiency of decision making units (DMUs) performing similar tasks in a production system that consumes multiple inputs to produce multiple outputs. So far, a number of DEA models have been developed: The CCR model, the BCC model and the FDH model are well known as basic DEA models. These models based on the domination structure in primal form are characterized by how to determine the production possibility set from a viewpoint of dual form; the convex cone, the convex hull and the free disposable hull for the observed data, respectively.

In this study, we suggest a model called generalized DEA (GDEA) model, which can treat the above stated basic DEA models in a unified way. In addition, by establishing the theoretical properties on relationships among the GDEA model and those DEA models, we prove that the GDEA model makes it possible to calculate the efficiency of DMU incorporating various preference structures of decision makers. Furthermore, we propose a dual approach to GDEA, $GDEA_D$ and also show that $GDEA_D$ can reveal domination relations among all DMUs.

Keywords: Data envelopment analysis; Generalized DEA; Efficient frontier

1. Introduction

Data envelopment analysis (DEA) was suggested by Charnes, Cooper and Rhodes (CCR), and built on the idea of Farrell [10] which is concerned with the estimation of technical efficiency and efficient frontiers. The CCR model [5,6] generalized the single output/single input ratio efficiency measure for each decision making unit (DMU) to multiple outputs/multiple inputs situations by forming the ratio of a weighted sum of outputs to a weighted sum of inputs. DEA is a method for measuring the relative efficiency of DMUs performing similar tasks in a production system that consumes multiple inputs to produce multiple outputs. The main characteristics of DEA are that (i) it can be applied to analyze multiple outputs and

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multiple inputs without preassigned weights, (ii) it can be used for measuring a relative efficiency based on the observed data without knowing information on the production function and (iii) decision makers' preferences can be incorporated in DEA models. Later, Banker, Charnes and Cooper (BCC) suggested a model for estimating technical efficiency and scale inefficiency in DEA. The BCC model [2] relaxed the constant returns to scale assumption of the CCR model and made it possible to investigate whether the performance of each DMU was conducted in region of increasing, constant or decreasing returns to scale in multiple outputs and multiple inputs situations. In addition, Tulkens [20] introduced a relative efficiency to non-convex free disposable hull (FDH) of the observed data defined by Deprins et al. [9], and formulated a mixed integer programming to calculate the relative efficiency for each DMU. Besides basic models as mentioned in the above, a number of extended models have been studied, for example, cone ratio model [8], polyhedral cone ratio model [7], Seiford and Thrall's model [16], Wei and Yu's model [21], and so on.

On the other hand, relationships between DEA and multiple criteria decision analysis (MCDA) have been studied from several viewpoints by many authors. Belton [3], and Belton and Vickers [4] measured an efficiency as a weighted sum of input and output. Stewart [17] showed the equivalence between the CCR model and some linear value function model for multiple outputs and multiple inputs. Joro et al. [13] proved structural correspondences between DEA models and multiple objective linear programming using an achievement scalarizing function proposed by Wierzbicki [22]. Especially, various ways of introducing preference information into DEA formulations have been developed. Golany [11] suggested a so-called target setting model, which allows decision makers to select the preferred set of output levels given the input levels of a DMU. Thanassoulis and Dyson [19] introduced models that can be used to estimate alternative output and input levels, in order to render relatively inefficient DMUs efficient. Zhu [23] proposed a model that calculates efficiency scores incorporating the decision makers' preference informations, whereas Korhonen [14] applied an interactive technique to progressively reveal preferences. Halme et al. [12] evaluated an efficiency of DMU in terms of pseudo-concave value function, by considering a tangent cone of the feasible set at the most preferred solution of decision maker. Agrell and Tind [1] showed correspondences among the CCR model [5], the BCC model [2] and the FDH model [20] and MCDA model according to the property of a partial Lagrangean relaxation. Yun et al. [24] suggested a concept of "value free efficiency" in the observed data.

In this study, we propose a generalized model for DEA, so-called GDEA model, which can treat basic DEA models, specifically, the CCR model, the BCC model and the FDH model in a unified way. In addition, we show theoretical properties on relationships among the GDEA model and those DEA models, and the GDEA model makes it possible to calculate the efficiency of DMUs incorporating various preference structures of decision makers. Finally, we suggest a dual approach GDEA_D to GDEA and show also that GDEA_D can reveal domination relations among all DMUs.

2. Basic DEA models

In the following discussion, we assume that there exist n DMUs to be evaluated. Each DMU consumes varying amounts of m different inputs to produce p different outputs. Specifically, DMUj consumes amounts $\boldsymbol{x}_j := (x_{ij})$ of inputs $(i = 1, \ldots, m)$ and produces amounts $\boldsymbol{y}_j := (y_{kj})$ of outputs $(k = 1, \ldots, p)$. For these constants, which generally take the form of observed data, we assume $x_{ij} > 0$ for each $i = 1, \ldots, m$ and $y_{kj} > 0$ for each $k = 1, \ldots, p$. Further, we assume that there are no duplicated units in the observed data. The $p \times n$ output matrix for the n DMUs is denoted by $\boldsymbol{Y}$, and the $m \times n$ input matrix for the n DMUs is denoted by $\boldsymbol{X}$. $\boldsymbol{x}_o := (x_{1o}, \ldots, x_{mo})$ and $\boldsymbol{y}_o := (y_{1o}, \ldots, y_{po})$ are amounts of inputs and outputs of DMUo, which is evaluated. In addition, ε is a small positive number (non-Archimedean) and $\mathbf{1} = (1, \ldots, 1)$ is a unit vector.

For convenience, the following notations for vectors in $\mathbb{R}^{p+m}$ will be used:

$$z_o > z_j \iff z_{io} > z_{ij}, \quad i = 1, \ldots, p+m,$$
$$z_o \geqq z_j \iff z_{io} \geqq z_{ij}, \quad i = 1, \ldots, p+m,$$
$$z_o \geqslant z_j \iff z_{io} \geqq z_{ij}, \quad i = 1, \ldots, p+m \quad \text{but } z_o \neq z_j.$$

So far, a number of DEA models have been developed. Among them, the CCR model [5,6], the BCC model [2] and the FDH model [20] are well known as basic DEA models. These models are based on the domination structure in primal form, and moreover these are characterized by how to determine the production possibility set from a viewpoint of dual form; the convex cone, the convex hull and the free disposable hull for the observed data, respectively.

2.1. The CCR model

The CCR model, which was suggested by Charnes et al. [5], is a fractional linear programming problem and can be solved by being transformed into an equivalent linear programming one. Therefore, the primal problem (CCR) with an input oriented model [1] can be formulated as the following:

$$\begin{aligned} \underset{\mu_k, v_i}{\text{maximize}} \quad & \sum_{k=1}^{p} \mu_k y_{ko} \qquad \text{(CCR)} \\ \text{subject to} \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & \sum_{k=1}^{p} \mu_k y_{kj} - \sum_{i=1}^{m} v_i x_{ij} \leqq 0, \quad j = 1, \ldots, n, \\ & \mu_k \geqq \varepsilon, \quad v_i \geqq \varepsilon, \quad k = 1, \ldots, p; \; i = 1, \ldots, m. \end{aligned}$$

The dual problem ($\mathrm{CCR_D}$) to the problem (CCR) is given by

$$\begin{aligned} \underset{\theta, \boldsymbol{\lambda}, \boldsymbol{s}_x, \boldsymbol{s}_y}{\text{minimize}} \quad & \theta - \varepsilon(\mathbf{1}^{\mathrm{T}} \boldsymbol{s}_x + \mathbf{1}^{\mathrm{T}} \boldsymbol{s}_y) \qquad (\mathrm{CCR_D}) \\ \text{subject to} \quad & \boldsymbol{X}\boldsymbol{\lambda} - \theta \boldsymbol{x}_o + \boldsymbol{s}_x = \mathbf{0}, \\ & \boldsymbol{Y}\boldsymbol{\lambda} - \boldsymbol{y}_o - \boldsymbol{s}_y = \mathbf{0}, \\ & \boldsymbol{\lambda} \geqq \mathbf{0}, \quad \boldsymbol{s}_x \geqq \mathbf{0}, \quad \boldsymbol{s}_y \geqq \mathbf{0}, \\ & \theta \in \mathbb{R}, \quad \boldsymbol{\lambda} \in \mathbb{R}^n, \quad \boldsymbol{s}_x \in \mathbb{R}^m, \quad \boldsymbol{s}_y \in \mathbb{R}^p. \end{aligned}$$

The ‘efficiency’ in the CCR model is introduced as follows (Fig. 1):

Definition 1 (*CCR-efficiency*). A DMU*o* is *CCR-efficient* if and only if the optimal value $\sum_{k=1}^{p} \mu_k^* y_{ko}$ to the problem (CCR) equals one. Otherwise, the DMU*o* is said to be *CCR-inefficient*.

Definition 2 (*$\mathrm{CCR_D}$-efficiency*). A DMU*o* is $\mathrm{CCR_D}$-*efficient* if and only if for the optimal solution $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ to the problem ($\mathrm{CCR_D}$), the following two conditions are satisfied:

(i) θ^* is equal to one;
(ii) the slack variables $\boldsymbol{s}_x^*$ and $\boldsymbol{s}_y^*$ are all zero.

Otherwise, the DMU*o* is $\mathrm{CCR_D}$-*inefficient*.

[1] In this paper, we deal with only the input oriented model for simplicity to condense the text.

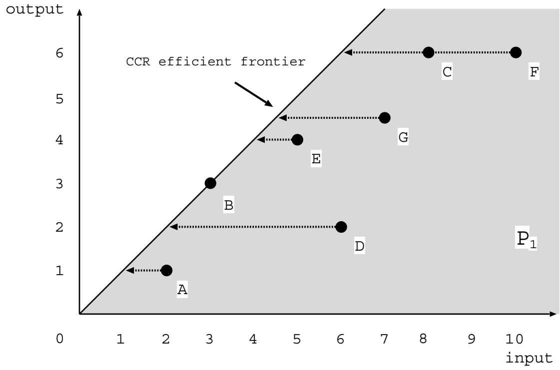

Fig. 1. CCR efficient frontier and production possibility set generated by the CCR model from the observed data.

Note that the above two definitions are evidently equivalent.

Additionally, the production possibility set P_1 in the CCR model is the *convex cone* (or conical hull) generated by the observed data, since one takes a viewpoint of the fact that the scale efficiency of a DMU is constant, that is to say, constant returns to scale. Therefore, P_1 can be denoted by

$$P_1 = \{(\boldsymbol{y}, \boldsymbol{x}) \mid Y\boldsymbol{\lambda} \geqq \boldsymbol{y}, X\boldsymbol{\lambda} \leqq \boldsymbol{x}, \boldsymbol{\lambda} \geqq \boldsymbol{0}\}$$

and the definition of CCR-efficiency (or CCR_D-efficiency) can be transformed into the following.

Definition 3. DMUo is said to be *Pareto efficient in* P_1 if and only if there does not exist $(\boldsymbol{y}, \boldsymbol{x}) \in P_1$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geqslant (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

The above definition will be used in Section 4.

2.2. The BCC model

The BCC model of Banker et al. [2] is formulated similarly to that for the CCR model. The dual problem for the BCC model is obtained by adding the convexity constraint $\boldsymbol{1}^\text{T}\boldsymbol{\lambda} = 1$ to the dual problem (CCR_D) and thus, the variable u_o appears in the primal problem. The efficiency degree of a DMUo with respect to the BCC model can be measured by solving the problem (Fig. 2)

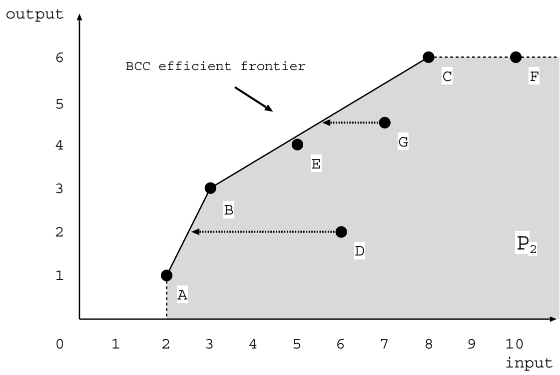

Fig. 2. BCC efficient frontier and production possibility set generated by the BCC model from the observed data.

$$\begin{aligned} \underset{\mu_k, v_i, u_o}{\text{maximize}} \quad & \sum_{k=1}^{p} \mu_k y_{ko} - u_o \\ \text{subject to} \quad & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & \sum_{k=1}^{p} \mu_k y_{kj} - \sum_{i=1}^{m} v_i x_{ij} - u_o \leqq 0, \quad j = 1, \ldots, n, \\ & \mu_k \geqq \varepsilon, \quad v_i \geqq \varepsilon, \quad k = 1, \ldots, p; \; i = 1, \ldots, m. \end{aligned} \tag{BCC}$$

The dual problem (BCC_D) to the problem (BCC) is formulated as follows:

$$\begin{aligned} \underset{\theta, \boldsymbol{\lambda}, \boldsymbol{s}_x, \boldsymbol{s}_y}{\text{minimize}} \quad & \theta - \varepsilon(\mathbf{1}^\text{T}\boldsymbol{s}_x + \mathbf{1}^\text{T}\boldsymbol{s}_y) \\ \text{subject to} \quad & \boldsymbol{X\lambda} - \theta\boldsymbol{x}_o + \boldsymbol{s}_x = \mathbf{0}, \\ & \boldsymbol{Y\lambda} - \boldsymbol{y}_o - \boldsymbol{s}_y = \mathbf{0}, \\ & \mathbf{1}^\text{T}\boldsymbol{\lambda} = 1, \\ & \boldsymbol{\lambda} \geqq \mathbf{0}, \quad \boldsymbol{s}_x \geqq \mathbf{0}, \quad \boldsymbol{s}_y \geqq \mathbf{0}, \\ & \theta \in \mathbb{R}, \quad \boldsymbol{\lambda} \in \mathbb{R}^n, \quad \boldsymbol{s}_x \in \mathbb{R}^m, \quad \boldsymbol{s}_y \in \mathbb{R}^p. \end{aligned} \tag{BCC_D}$$

The definition of 'efficiency' in the BCC model is given as follows, and the two definitions are equivalent.

Definition 4 (*BCC-efficiency*). A DMU*o* is *BCC-efficient* if and only if the optimal value $(\sum_{k=1}^{p} \mu_k^* y_{ko} - u_o^*)$ to the problem (BCC) equals one. Otherwise, the DMU*o* is said to be *BCC-inefficient*.

Definition 5 (*BCC_D-efficiency*). A DMU*o* is BCC_D-*efficient* if and only if for an optimal solution $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ to the problem (BCC_D), the following two conditions are satisfied:

(i) θ^* is equal to one;
(ii) the slack variables $\boldsymbol{s}_x^*$ and $\boldsymbol{s}_y^*$ are all zero.

Otherwise, the DMU*o* is said to be BCC_D-*inefficient*.

The presence of the constraint $\mathbf{1}^\text{T}\boldsymbol{\lambda} = 1$ in the dual problem (BCC_D) yields that the production possibility set P_2 in the BCC model is the *convex hull* generated by the observed data. Therefore, P_2 can be obtained as

$$P_2 = \{(\boldsymbol{y}, \boldsymbol{x}) \mid Y\boldsymbol{\lambda} \geqq \boldsymbol{y}, X\boldsymbol{\lambda} \leqq \boldsymbol{x}, \mathbf{1}^\text{T}\boldsymbol{\lambda} = 1, \boldsymbol{\lambda} \geqq \mathbf{0}\}$$

and the definition of BCC-efficiency (or BCC_D-efficiency) can be transformed into the following:

Definition 6. DMU*o* is said to be *Pareto efficient in* P_2 if and only if there does not exist $(\boldsymbol{y}, \boldsymbol{x}) \in P_2$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geqslant (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

The above definition will be used in Section 4.

2.3. The FDH model

Adding the constraints $\lambda_j \in \{0, 1\}$ for each $j = 1, \ldots, n$, to the problem (BCC_D), the FDH model by Tulkens [20] is formulated as follows:

$$\begin{aligned}
\underset{\theta,\boldsymbol{\lambda},\boldsymbol{s}_x,\boldsymbol{s}_y}{\text{minimize}} \quad & \theta - \varepsilon(\mathbf{1}^{\mathrm{T}}\boldsymbol{s}_x + \mathbf{1}^{\mathrm{T}}\boldsymbol{s}_y) \\
\text{subject to} \quad & X\boldsymbol{\lambda} - \theta\boldsymbol{x}_o + \boldsymbol{s}_x = \mathbf{0}, \\
& Y\boldsymbol{\lambda} - \boldsymbol{y}_o - \boldsymbol{s}_y = \mathbf{0}, \\
& \mathbf{1}^{\mathrm{T}}\boldsymbol{\lambda} = 1; \quad \lambda_j \in \{0,1\} \quad \text{for each } j = 1,\ldots,n, \\
& \boldsymbol{\lambda} \geqq \mathbf{0}, \quad \boldsymbol{s}_x \geqq \mathbf{0}, \quad \boldsymbol{s}_y \geqq \mathbf{0}, \\
& \theta \in \mathbb{R}, \quad \boldsymbol{\lambda} \in \mathbb{R}^n, \quad \boldsymbol{s}_x \in \mathbb{R}^m, \quad \boldsymbol{s}_y \in \mathbb{R}^p.
\end{aligned} \tag{FDH$_\mathrm{D}$}$$

However, here it is seen that the problem ($\mathrm{FDH_D}$) is a mixed integer programming problem, and hence the traditional linear optimization methods cannot apply to it. An optimal solution is obtained by means of a simple vector comparison procedure to the end.

For a DMUo, the optimal solution θ^* to the problem ($\mathrm{FDH_D}$) is equal to the value R_o^* defined by

$$R_o^* = \min_{j \in D(o)} \max_{i=1,\ldots m} \left\{ \frac{x_{ij}}{x_{io}} \right\}, \tag{1}$$

where $D(o) = \{j \mid \boldsymbol{x}_j \leqslant \boldsymbol{x}_o \text{ and } \boldsymbol{y}_j \geqq \boldsymbol{y}_o, j = 1,\ldots,n\}$.

R_o^* is substituted for θ^* as the efficiency degree for DMUo in the FDH model. Also, the 'efficiency' in the FDH model is given in the following.

Definition 7 (*FDH-efficiency*). A DMUo is *FDH-efficient* if and only if R_o^* equals to one. If $R_o^* < 1$, the DMUo is said to be *FDH-inefficient*.

Definition 8 (*FDH*$_\mathrm{D}$*-efficiency*). A DMUo is $\mathrm{FDH_D}$-*efficient* if and only if for an optimal solution $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_x^*, \boldsymbol{s}_y^*)$ to the problem ($\mathrm{FDH_D}$), the following two conditions are satisfied (Fig. 3):

(i) θ^* is equal to one;
(ii) the slack variables $\boldsymbol{s}_x^*$ and $\boldsymbol{s}_y^*$ are all zero.

Otherwise, the DMUo is said to be $\mathrm{FDH_D}$-*inefficient*.

The above two definitions are equivalent forms, and the production possibility set P_3, which is a free disposable hull, is given by

$$P_3 = \{(\boldsymbol{y},\boldsymbol{x}) \mid Y\boldsymbol{\lambda} \geqq \boldsymbol{y}, X\boldsymbol{\lambda} \leqq \boldsymbol{x}, \mathbf{1}^{\mathrm{T}}\boldsymbol{\lambda} = 1, \lambda_j \in \{0,1\}, j = 1,\ldots,n\}. \tag{2}$$

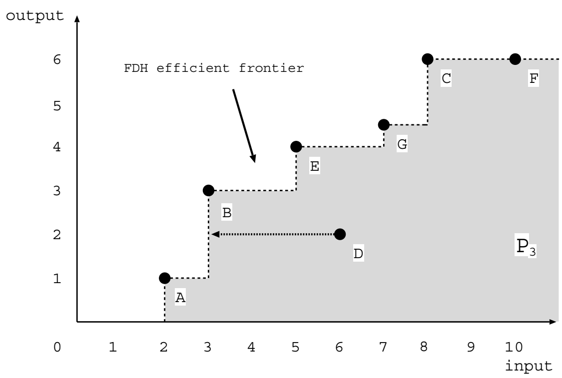

Fig. 3. FDH efficient frontier and production possibility set generated by the FDH model from the observed data.

Besides, the definition of FDH-efficiency (or FDH_D-efficiency) can be transformed into the following:

Definition 9. DMUo is said to be *Pareto efficient in* P_3 if and only if there does not exist $(\boldsymbol{y},\boldsymbol{x}) \in P_3$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geqslant (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

The above definition will be used in Section 4.

3. GDEA based on parametric domination structure

In this section, we formulate the GDEA model based on a domination structure and define a new 'efficiency' in the GDEA model. Next, we establish relationships between the GDEA model and basic DEA models mentioned in Section 2.

Now, we formulate a generalized DEA model by employing the augmented Tchebyshev scalarizing function [15]. The GDEA model, which can evaluate the efficiency in several basic models as special cases, is the following:

$$\begin{aligned} \underset{\Delta,\mu_k,\nu_i}{\text{maximize}} \quad & \Delta \qquad\qquad \text{(GDEA)} \\ \text{subject to} \quad & \Delta \leqq \tilde{d}_j + \alpha\left(\sum_{k=1}^{p}\mu_k(y_{ko}-y_{kj}) + \sum_{i=1}^{m}\nu_i(-x_{io}+x_{ij})\right), \quad j=1,\dots,n, \\ & \sum_{k=1}^{p}\mu_k + \sum_{i=1}^{m}\nu_i = 1, \\ & \mu_k, \nu_i \geqq \varepsilon, \quad k=1,\dots,p; \ i=1,\dots,m, \end{aligned}$$

where $\alpha > 0$ is appropriately given according to given problems, and $\tilde{d}_j (j = 1,\dots,n)$ is the value of multiplying the maximal component of $(y_{1o} - y_{1j},\dots,y_{po} - y_{pj}, -x_{1o} + x_{1j},\dots,-x_{mo} + x_{mj})$ by its corresponding weight. (For example, if $(y_{1o} - y_{1j}, -x_{1o} + x_{1j}) = (2, -1)$, then $\tilde{d}_j = 2\mu_1$.)

Note that when $j = o$, the right-hand side of the inequality constraint in the problem (GDEA) is zero, and hence its optimal value is not greater than zero. We define 'efficiency' in the GDEA model as follows.

Definition 10 (*α-efficiency*). For a given positive number α, DMUo is defined to be α-*efficient* if and only if the optimal value to the problem (GDEA) is equal to zero. Otherwise, DMUo is said to be α-*inefficient*.

3.1. Relationships between GDEA and DEA

In this subsection, we establish theoretical properties on relationships among efficiencies in the basic DEA models and that in the GDEA model.

Theorem 1. *DMUo is FDH-efficient if and only if DMUo is α-efficient for some sufficiently small positive number α.*

Proof (*only if part*). Let Δ^*, $(\mu_1^*,\dots,\mu_p^*)$ and $(\nu_1^*,\dots,\nu_m^*)$ be the optimal solution for the DMUo. Negate that DMUo is α-efficient for some sufficiently small positive α. Then for any sufficiently small positive α, $\Delta^* < 0$, that is,

$$\tilde{d}_j + \alpha\left(\sum_{k=1}^{p}\mu_k^*(y_{ko}-y_{kj}) + \sum_{i=1}^{m}\nu_i^*(-x_{io}+x_{ij})\right) < 0 \quad \text{for some } j \neq o. \tag{3}$$

The necessary and sufficient condition so that the above inequality (3) holds for any sufficiently small positive α is that

$$\tilde{d}_j = \max_{\substack{k=1,\dots,p \\ i=1,\dots,m}} \{\mu_k^*(y_{ko} - y_{kj}), v_i^*(-x_{io} + x_{ij})\} < 0 \tag{4}$$

and since $(\mu_1^*, \dots, \mu_p^*)$ and $(v_1^*, \dots, v_m^*)$ are strictly positive, the inequality (4) implies that $(\boldsymbol{y}_j, -\boldsymbol{x}_j) > (\boldsymbol{y}_o, -\boldsymbol{x}_o)$ for some $j \neq o$. Thus, $j \in D(o)$ and $\max_{i=1,\dots,m}\{x_{ij}/x_{io}\} < 1$, which means that $R_o^* = \min_{j\in D(o)} \max_{i=1,\dots,m}\{x_{ij}/x_{io}\} < 1$. This contradicts the assumption that DMU*o* is FDH-efficient, and therefore DMU*o* is α-efficient for some sufficiently small positive α.

(*if part*) Suppose that DMU*o* is FDH-inefficient. Then $R_o^* < 1$, which yields that there exists some $j \in D(o) = \{j \mid \boldsymbol{x}_j \leqslant \boldsymbol{x}_o \text{ and } \boldsymbol{y}_j \geqq \boldsymbol{y}_o, j = 1, \dots, n\}$ such that $\max_{i=1,\dots,m}\{x_{ij}/x_{io}\} < 1$. Thus, $\boldsymbol{y}_j \geqq \boldsymbol{y}_o$ and $\boldsymbol{x}_j < \boldsymbol{x}_o$ for such a j. For any positive $(\mu_1, \dots, \mu_p)$ and $(v_1, \dots, v_m)$, we have

$$\mu_k(y_{ko} - y_{kj}) \leqq 0, \quad k = 1, \dots, p \quad \text{and} \quad v_i(-x_{io} + x_{ij}) < 0, \; i = 1, \dots, m. \tag{5}$$

From inequalities of the above (5), the following inequalities hold:

$$\tilde{d}_j = \max_{\substack{k=1,\dots,p \\ i=1,\dots,m}} \{\mu_k(y_{ko} - y_{kj}), v_i(-x_{io} + x_{ij})\} \leqq 0 \tag{6}$$

and

$$\left(\sum_{k=1}^{p} \mu_k(y_{ko} - y_{kj}) + \sum_{i=1}^{m} v_i(-x_{io} + x_{ij}) \right) < 0. \tag{7}$$

Multiplying (7) by any positive α and adding to (6) yields that

$$\tilde{d}_j + \alpha\left(\sum_{k=1}^{p} \mu_k(y_{ko} - y_{kj}) + \sum_{i=1}^{m} v_i(-x_{io} + x_{ij}) \right) < 0 \quad \text{for some } j,$$

which is a contradiction to the α-efficiency for some sufficiently small positive α. Hence, it has been shown that the DMU*o* is FDH-efficient. □

Theorem 2. *DMUo is BCC-efficient if and only if DMUo is α-efficient for some sufficiently large positive number α.*

Consider the problem (GDEA) in which the constraint $\sum_{k=1}^{p} \mu_k y_{ko} = \sum_{i=1}^{m} v_i x_{io}$ is added to the problem (GDEA):

$$\begin{aligned}
&\underset{\Delta,\mu_k,v_i}{\text{maximize}} \quad \Delta && \text{(GDEA)} \\
&\text{subject to} \quad \Delta \leqq \tilde{d}_j + \alpha\left(\sum_{k=1}^{p} \mu_k(y_{ko} - y_{kj}) + \sum_{i=1}^{m} v_i(-x_{io} + x_{ij}) \right), \quad j = 1, \dots, n, \\
&\qquad \sum_{k=1}^{p} \mu_k y_{ko} - \sum_{i=1}^{m} v_i x_{io} = 0, \\
&\qquad \sum_{k=1}^{p} \mu_k + \sum_{i=1}^{m} v_i = 1, \\
&\qquad \mu_k, v_i \geqq \varepsilon, \quad k = 1, \dots, p; \; i = 1, \dots, m.
\end{aligned}$$

Theorem 3. *DMUo is CCR-efficient if and only if DMUo is α-efficient for some sufficiently large positive α when regarding the problem* (GDEA) *as the problem* (GDEA).

The proof of Theorems 2 and 3 follows along the lines of the one of Theorem 1. From the stated theorems, it is seen that the FDH-efficiency, the BCC-efficiency and the CCR-efficiency for each DMU can be evaluated by varying the parameter α in the problem (GDEA) from a sufficiently small value to a sufficiently large one. It cannot be known a prior how small/large is sufficiently small/large, and hence the value of α is empirically given. What is important is to see how the efficiency of each DMU changes by varying the value of α.

3.2. An illustrative example

Here, we explain the α-efficiency in the GDEA model with a simple illustrative example and reveal domination relations among all DMUs by GDEA.

Assume that there are six DMUs which consume one input to produce one output, as seen in Table 1.

Table 2 shows the results of efficiency in the basic DEA models and α-efficiency in the GDEA model. In the upper half part of Table 2, we see that a DMU is efficient if the optimal value is equal to one in the CCR model, the BCC model and the FDH models, respectively. The lower half part of Table 2 shows the α-efficiency by changing a parameter α. It can be seen that if $\alpha = 0.1$, the α-efficiency of each DMU is the same as the FDH-efficiency. If $\alpha = 10$, the α-efficiency of each DMU is the same as the BCC-efficiency, and moreover if $\alpha = 10$ in the problem (GDEA), then the α-efficiency is equivalent to the CCR-efficiency. Furthermore, Figs. 4–6 represent the efficient frontier generated by varying α in the GDEA model.

Through this example, it was shown that by varying the value of parameter α, various efficiency of the basic DEA models can be measured in a unified way on the basis of this GDEA model, and furthermore the relationships among efficiency for these models become transparent.

Table 1
An example of 1-input and 1-output

DMU	*A*	*B*	*C*	*D*	*E*	*F*
Input	2	3	4.5	4	6	5.5
Output	1	3	3.5	2	5	4

Table 2
The optimal values in basic DEA models and GDEA model

DMU	*A*	*B*	*C*	*D*	*E*	*F*
CCR model	0.50	**1.00**	0.78	0.50	0.83	0.73
BCC model	**1.00**	**1.00**	0.83	0.63	**1.00**	0.75
FDH model	**1.00**	**1.00**	**1.00**	0.75	**1.00**	**1.00**
(i) $\alpha = 10$ (GDEA)	−9.33	**0.00**	−3.25	−11.33	−0.73	−3.74
(ii) $\alpha = 10$	**0.00**	**0.00**	−2.10	−11.00	**0.00**	−3.35
(iii) $\alpha = 3$	**0.00**	**0.00**	**0.00**	−4.00	**0.00**	−0.55
(iv) $\alpha = 1$	**0.00**	**0.00**	**0.00**	−2.00	**0.00**	**0.00**
(v) $\alpha = 0.1$	**0.00**	**0.00**	**0.00**	−1.10	**0.00**	**0.00**

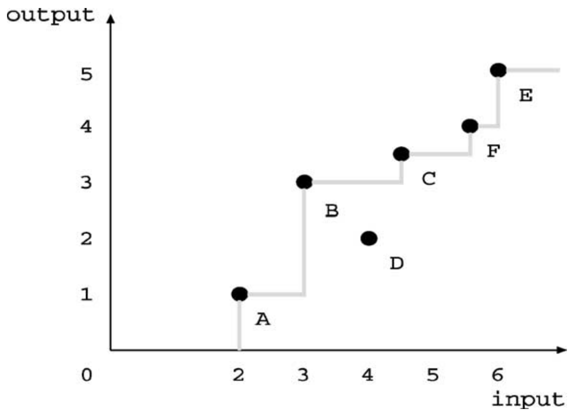

Fig. 4. Efficient frontier generated by the GDEA model with $\alpha = 0$.

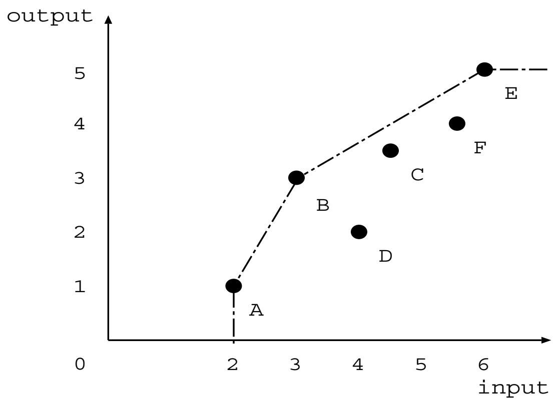

Fig. 5. Efficient frontier generated by the GDEA model with $\alpha = 10$.

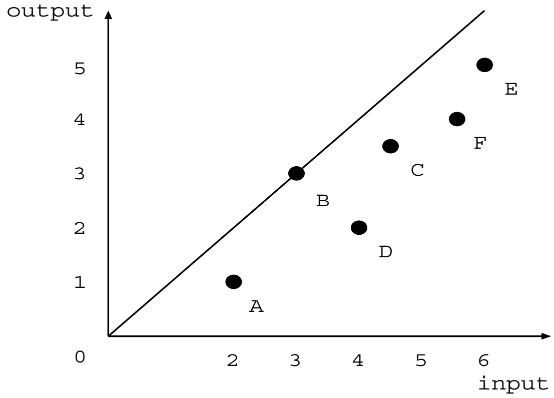

Fig. 6. Efficient frontier generated by the GDEA′ model with $\alpha = 10$.

4. GDEA based on production possibility

In this section, we consider a dual approach to GDEA introduced in Section 3. We formulate the GDEA_D model based on the production possibility set and define 'efficiency' in the GDEA_D model. Next, we establish relationships between the GDEA_D model and dual models to basic DEA models mentioned in Section 2.

To begin with, an output–input vector $\boldsymbol{z}_j$ of a DMU$j, j = 1, \ldots, n$, and output–input matrix Z of all DMUs respectively, denoted by

$$\boldsymbol{z}_j := \begin{pmatrix} \boldsymbol{y}_j \\ -\boldsymbol{x}_j \end{pmatrix}, \quad j = 1, \ldots, n, \quad \text{and} \quad Z := \begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{X} \end{pmatrix}.$$

In addition, we denote a $(p+m) \times n$ matrix Z_o by $Z_o := (\boldsymbol{z}_o, \ldots, \boldsymbol{z}_o)$, where o is the index of DMU to be evaluated.

The production possibility sets in the CCR model, the BCC model and the FDH model in Section 2 are reformulated as follows:

$$\begin{aligned} P_1' &= \{\boldsymbol{z} \mid Z\boldsymbol{\lambda} \geqq \boldsymbol{z}, \boldsymbol{\lambda} \geqq \boldsymbol{0}\}, \\ P_2' &= \{\boldsymbol{z} \mid Z\boldsymbol{\lambda} \geqq \boldsymbol{z}, \boldsymbol{1}^{\mathrm{T}}\boldsymbol{\lambda} = 1, \boldsymbol{\lambda} \geqq \boldsymbol{0}\}, \\ P_3' &= \{\boldsymbol{z} \mid Z\boldsymbol{\lambda} \geqq \boldsymbol{z}, \boldsymbol{1}^{\mathrm{T}}\boldsymbol{\lambda} = 1, \lambda_j \in \{0,1\}, j = 1, \ldots, n\} \end{aligned}$$

and the 'efficiencies' in these models are redefined.

Definition 11. DMUo is said to be *Pareto efficient in* P_1' if and only if there does not exist $(\boldsymbol{y}, -\boldsymbol{x}) \in P_1'$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geqslant (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

Definition 12. DMUo is said to be *Pareto efficient in* P_2' if and only if there does not exist $(\boldsymbol{y}, -\boldsymbol{x}) \in P_2'$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geqslant (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

Definition 13. DMUo is said to be *Pareto efficient in* P_3' if and only if there does not exist $(\boldsymbol{y}, -\boldsymbol{x}) \in P_3'$ such that $(\boldsymbol{y}, -\boldsymbol{x}) \geqslant (\boldsymbol{y}_o, -\boldsymbol{x}_o)$.

Remark 1 [13]. Here, the Definitions 11–13 are corresponding to the CCR-efficiency (or CCR_D-efficiency), BCC-efficiency (or BCC_D-efficiency) and the FDH-efficiency (or FDH_D-efficiency), respectively.

Here, dual problem to the problem (GDEA) introduced in Section 3 is formulated as follows:

$$\begin{aligned} \underset{\omega,\kappa,\boldsymbol{\lambda},\boldsymbol{s}_z}{\text{minimize}} \quad & \omega - \varepsilon \boldsymbol{1}^{\mathrm{T}}\boldsymbol{s}_z \qquad (\text{GDEA}_\text{D}) \\ \text{subject to} \quad & \{\alpha(Z_o - Z) + D_z\}\boldsymbol{\lambda} - \boldsymbol{\omega} + \boldsymbol{s}_z + \kappa \boldsymbol{z}_o = \boldsymbol{0}, \\ & \boldsymbol{1}^{\mathrm{T}}\boldsymbol{\lambda} = 1, \\ & \boldsymbol{\lambda} \geqq \boldsymbol{0}, \ \boldsymbol{s}_z \geqq \boldsymbol{0}, \end{aligned}$$

where $\boldsymbol{\omega} = (\omega, \ldots, \omega)$ and α is a given positive number. A $(p+m) \times n$ matrix $D_z := (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n)$ is a matrix $(Z_o - Z)$ replaced by 0, except for the maximal component (if there exist plural maximal components, only one is chosen from among them) in each column. Especially, it is seen that when κ is fixed at 0, the problem (GDEA_D) becomes the dual problem to the problem (GDEA), since κ is the dual variable to the second constraint in the problem (GDEA).

We define an 'efficiency' for a DMUo in the GDEA$_{\mathrm{D}}$ model.

Definition 14 (α_{D}-*efficiency*). For a given positive α, DMUo is said to be α_{D}-*efficient* if and only if the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem (GDEA$_{\mathrm{D}}$) satisfies the following two conditions:

(i) ω^* is equal to zero;
(ii) the slack variable $\boldsymbol{s}_z^*$ is zero.

Otherwise, DMUo is said to be α_{D}-*inefficient*.

We, particularly, note that for an optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem GDEA$_{\mathrm{D}}$, ω^* is not greater than zero because of the strong duality of (GDEA) and (GDEA$_{\mathrm{D}}$) (in linear programming problem), and 'non-Archimedean' property of ε.

4.1. Relationships between GDEA$_D$ and DEA

In this subsection, we establish theoretical properties on relationships among efficiencies in basic DEA models and the GDEA$_{\mathrm{D}}$ model.

Theorem 4. *Let κ be fixed at* 0 *in the problem* (GDEA$_{\mathrm{D}}$). *DMUo is Pareto efficient in P_3' if and only if DMUo is α_{D}-efficient for some sufficiently small positive number α.*

Proof (*only if part*). Assume that DMUo is Pareto efficient in P_3'. Then, there does not exist $\boldsymbol{\lambda}$ such that

$$\boldsymbol{z}_j = Z\boldsymbol{\lambda} \geqslant Z_o\boldsymbol{\lambda} = \boldsymbol{z}_o, \tag{8}$$

where $\boldsymbol{\lambda} \in \{\boldsymbol{\lambda} \mid \mathbf{1}^{\mathrm{T}}\boldsymbol{\lambda} = 1, \lambda_j \in \{0,1\}, j = 1,\ldots,n\}$.

Negate that DMUo is α_{D}-efficient for some sufficiently small positive α. Then, for an optimal solution $(\omega^*, \boldsymbol{s}_z^*, \boldsymbol{\lambda}^*)$ to the problem (GDEA$_{\mathrm{D}}$), $\omega^* < 0$ or $\boldsymbol{s}_z^* \geqslant 0$. In other words, for any sufficiently small positive α, the following inequality holds:

$$\{\alpha(Z_o - Z) + D_z\}\boldsymbol{\lambda}^* = \boldsymbol{\omega}^* - \boldsymbol{s}_z^* \leqslant \mathbf{0}. \tag{9}$$

The necessary condition that the inequality (9) holds for any sufficiently small positive α is that $D_z\boldsymbol{\lambda}^* \leqslant \mathbf{0}$. This implies $\boldsymbol{d}_j \leqslant 0$, for some j, since $\boldsymbol{\lambda}^* \geqslant 0$. Besides, from the definition of $\boldsymbol{d}_j$, we have $\boldsymbol{z}_o - \boldsymbol{z}_j \leqslant 0$. This is a contradiction to the inequality (8).

(*if part*) If DMUo is α_{D}-efficient for some sufficiently small positive α, then from the first constraint of the problem (GDEA$_{\mathrm{D}}$), the following equality is obtained:

$$\{\alpha(Z_o - Z) + D_z\}\boldsymbol{\lambda}^* = \boldsymbol{\omega}^* - \boldsymbol{s}_z^* = \mathbf{0}, \tag{10}$$

where $(\omega^*, \boldsymbol{s}_z^*, \boldsymbol{\lambda}^*)$ is an optimal solution to the problem (GDEA$_{\mathrm{D}}$).

Suppose that DMUo is not Pareto efficient in P_3'. Then there exists $\boldsymbol{z} \in P_3'$ such that $\boldsymbol{z} \geqslant \boldsymbol{z}_o$. This means that

$$\boldsymbol{z}_j = Z\hat{\boldsymbol{\lambda}} \geqslant Z_o\hat{\boldsymbol{\lambda}} = \boldsymbol{z}_o, \quad \hat{\boldsymbol{\lambda}} \in \{\mathbf{1}^{\mathrm{T}}\hat{\boldsymbol{\lambda}} = 1, \hat{\lambda}_j \in \{0,1\}, j = 1,\ldots,n\}. \tag{11}$$

From the expression (11), $\boldsymbol{d}_j \leqslant 0$ for $j : \hat{\lambda}_j = 1$ and $D_z\hat{\boldsymbol{\lambda}} \leqslant 0$. Multiplying the inequality of (11) by an arbitrary positive α and adding it to $D_z\hat{\boldsymbol{\lambda}}$ yields

$$\{\alpha(Z_o - Z) + D_z\}\hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\omega}} - \hat{\boldsymbol{s}}_z \leqslant 0$$

and thus, $(\hat{\omega}, \hat{\boldsymbol{s}}_z, \hat{\boldsymbol{\lambda}})$ is a feasible solution of the problem (GDEA$_{\mathrm{D}}$). However, this contradicts to the fact that the expression (10) holds for $(\omega^*, \boldsymbol{s}_z^*, \boldsymbol{\lambda}^*)$. □

The proofs of the following theorems follow along the lines of the one of Theorem 4.

Theorem 5. *Let κ be fixed at* 0 *in the problem* (GDEA$_D$). *DMUo is Pareto efficient in P_2' if and only if DMUo is α_D-efficient for some sufficiently large positive number α.*

Theorem 6. *DMUo is Pareto efficient in P_1' if and only if DMUo is α_D-efficient for some sufficiently large positive number α.*

4.2. Optimal solutions to (GDEA$_D$)

In this subsection, we explain the meaning of optimal solutions $\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^* = (s_y^*, s_x^*)$ to the problem (GDEA$_D$). ω^* gives the measurement of relative efficiency for DMU*o*. In other words, it represents the degree how inefficient DMU*o* is, that is, how far DMU*o* is from the efficient frontier generated with the given α. κ is a dual variable to the constraint $\sum_{k=1}^{p} \mu_k y_{ko} = \sum_{i=1}^{m} v_i x_{io}$ in the primal problem (GDEA) and therefore, we put $\kappa = 0$ when considering the FDH and the BCC efficiency. $\boldsymbol{\lambda}^* := (\lambda_1^*, \ldots, \lambda_n^*)$ represents a domination relation between DMU*o* and another DMUs. That is, it means that the DMU*o* is dominated by DMU*j* if λ_j for some $j \neq o$ is positive. $\boldsymbol{s}_x^*$ represents the slack of inputs and $\boldsymbol{s}_y^*$ does the surplus of outputs for performance of the DMU*o*.

Consider an illustrative example as shown in Table 3. Table 4 shows the results of the CCR-efficiency, BCC-efficiency and FDH-efficiency, respectively, in the example.

Table 5 shows the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem (GDEA$_D$) ($\varepsilon = 10^{-6}$) when α is given as 10^{-6} and κ is fixed at 0. Table 6 shows the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem (GDEA$_D$)

Table 3
An example of 1-input and 1-output

DMU	*A*	*B*	*C*	*D*	*E*	*F*	*G*
Input	2	3	8	6	5	10	7
Output	1	3	6	2	4	6	4

Table 4
Optimal value in the problems (CCR), (BCC) and (FDH)

DMU	CCR model	BCC model	FDH model
A	0.5	1	1
B	1	1	1
C	0.75	1	1
D	0.333	0.417	0.5
E	0.8	0.933	1
F	0.6	0.8	0.8
G	0.571	0.667	0.714

Table 5
Optimal solution to (GDEA$_D$) with $\alpha = 10^{-6}$ and fixed $\kappa = 0$

DMU	ω^*	$\boldsymbol{\lambda}^*$	$\boldsymbol{s}_z^* = (\boldsymbol{s}_y^*, \boldsymbol{s}_x^*)$
A	0	$\lambda_A^* = 1$	$(0, 0)$
B	0	$\lambda_B^* = 1$	$(0, 0)$
C	0	$\lambda_C^* = 1$	$(0, 0)$
D	−0.5	$\lambda_B^* = \lambda_E^* = 0.5$	$(0, 0)$
E	0	$\lambda_E^* = 1$	$(0, 0)$
F	0	$\lambda_C^* = 1$	$(0, 2)$
G	0	$\lambda_E^* = 1$	$(0, 2)$

Table 6
Optimal solution to ($GDEA_D$) with $\alpha = 10$ and fixed $\kappa = 0$

DMU	ω^*	$\boldsymbol{\lambda}^*$	$\boldsymbol{s}_z^* = (\boldsymbol{s}_y^*, \boldsymbol{s}_x^*)$
A	0	$\lambda_A^* = 1$	$(0, 0)$
B	0	$\lambda_B^* = 1$	$(0, 0)$
C	0	$\lambda_C^* = 1$	$(0, 0)$
D	−17.803	$\lambda_A^* = 0.765,\ \lambda_B^* = 0.235$	$(0, 0)$
E	−0.441	$\lambda_B^* = 0.631,\ \lambda_C^* = 0.369$	$(0, 0)$
F	0	$\lambda_C^* = 1$	$(0, 20)$
G	−8.281	$\lambda_B^* = 0.378,\ \lambda_C^* = 0.622$	$(0, 0)$

($\varepsilon = 10^{-6}$) when α is given by 10 and κ is fixed at 0. Finally, Table 7 shows the optimal solution $(\omega^*, \kappa^*, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem ($GDEA_D$) ($\varepsilon = 10^{-6}$) when α is given as 100. Also, we can see that the FDH-efficiency, BCC-efficiency and CCR-efficiency are equivalent to the α-efficiency, respectively, from the result of Tables 5–7 and Figs. 7–9. In other words, the FDH-efficiency, BCC-efficiency and CCR-efficiency can be obtained by changing the parameter α in the $GDEA_D$ model.

Now, we interpret the value of optimal solution $(\omega^*, \kappa, \boldsymbol{\lambda}^*, \boldsymbol{s}_z^*)$ to the problem ($GDEA_D$). ω^* gives the measurement of relative efficiency for DMU*o*, and DMU with non-zero ω^* is inefficient. Moreover, it represents the degree how inefficient DMU*o* is, that is, how far DMU*o* is from the efficient frontier generated with given α. For example, DMUs in Table 7 are arranged in the following order by their efficiencies:

Table 7
Optimal solution to ($GDEA_D$) with $\alpha = 100$ and non-fixed κ

DMU	ω^*	$\boldsymbol{\lambda}^*$	$\boldsymbol{s}_z^* = (\boldsymbol{s}_y^*, \boldsymbol{s}_x^*)$	κ^*
A	−131.333	$\lambda_C^* = 1$	(0, 0)	368.667
B	0	$\lambda_B^* = 1$	(0, 0)	0
C	−41.143	$\lambda_B^* = 1$	(0, 0)	−57.357
D	−249.5	$\lambda_C^* = 1$	(0, 0)	75.25
E	−32.778	$\lambda_B^* = 1$	(0, 0)	−33.444
F	−75.0	$\lambda_C^* = 1$	(0, 0)	−12.5
G	−90.545	$\lambda_C^* = 1$	(0, 0)	27.364

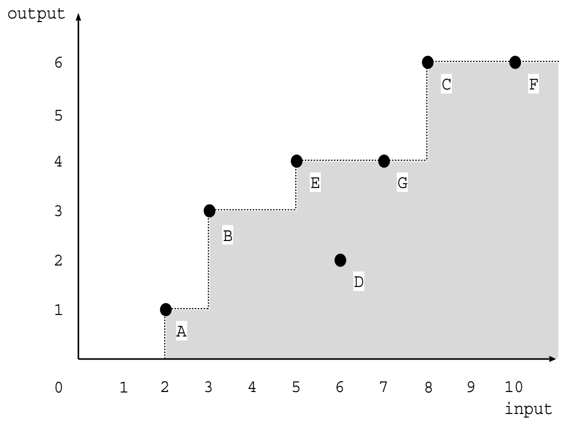

Fig. 7. Efficient frontier generated by $GDEA_D$ model with $\alpha = 10^{-6}$ and fixed $\kappa = 0$.

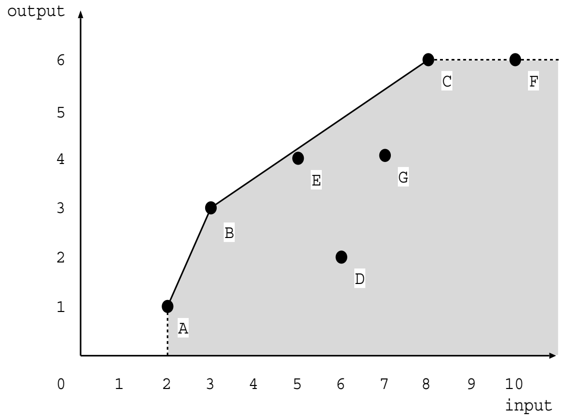

Fig. 8. Efficient frontier generated by $GDEA_D$ model with $\alpha = 10$ and fixed $\kappa = 0$.

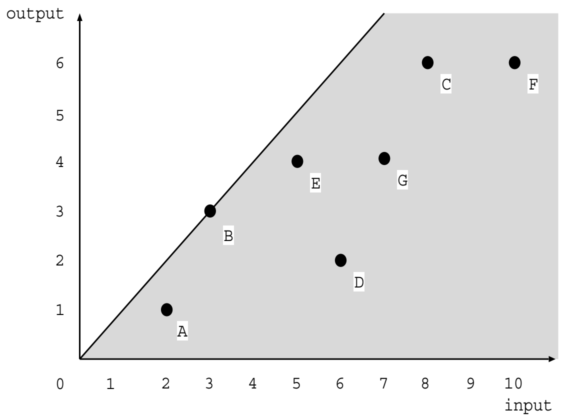

Fig. 9. Efficient frontier generated by $GDEA_D$ model with $\alpha = 100$ and non-fixed κ.

$$0 = B > E > C > F > G > A > D.$$

Thus, DMU B with the best efficiency is on the efficient frontier, but the worst DUM D is farthest from it.

As mentioned in the above, κ is a dual variable to the constraint $\sum_{k=1}^{p} \mu_k y_{ko} = \sum_{i=1}^{m} v_i x_{io}$ in the primal problem (GDEA) which can generates the efficiency equivalent to the CCR model. Thus, κ is not fixed in the case of obtaining the CCR efficiency.

$\boldsymbol{\lambda}^* := (\lambda_1^*, \ldots, \lambda_n^*)$ represents a domination relation between DMUo and another DMUs. That is, it means that the DMUo is dominated by DMUj if λ_j for some $j \neq o$ is positive. For example, as seen in Table 5, the optimal solution for the DMU D is $\lambda_B^* = 0.5$ and $\lambda_E^* = 0.5$, and hence DMU D is dominated by DMU B and DMU E (see Fig. 7). In addition, in Table 6, the optimal solution for the DMU E is $\lambda_B^* = 0.631$ and $\lambda_C^* = 0.369$, and hence DMU E is dominated by linear combination of DMU B and DMU C (see Fig. 8). [2]

[2] The domination set in the GDEA model does not necessarily agree with the reference one by the existing DEA models. The reference points themselves are of domination set, or a part of their linear combination is of domination set.

s_x^* represents the slack of inputs and s_y^* does the surplus of outputs for performance of the DMUo. For instance, in Table 5 DMU G has the optimal solution $\omega^* = 0, \lambda_E^* = 1$ and $s_x^* = 2$, and it is α-inefficient because s_x^* is not equal to zero although $\omega^* = 0$. It implies that DMU G has the larger surplus amount of input than DMU E with the same output.

5. Comparison between GDEA and DEA models

Now, we compare the efficiency in basic DEA models and the GDEA model for the data in Taylor et al. [18]. The data for thirteen Mexican commercial banks in two years (1990–1991) is from Taylor et al. [18]. As shown in Table 8, each bank has the total income as the single output. Total income is the sum of a bank's interest and non-interest income. Total deposits and total non-interest expense are the two inputs used to generate the output. Interest income includes interest earned from loan activities. Total non-interest income includes dividends, fees, and other non-interest revenue. The total deposits input variable includes the bank's interest paying deposit liabilities. Total non-interest expense includes personnel and administrative costs, commissions paid, banking support fund contributions and other non-interest operating costs. Thus, we evaluate the efficiency for each bank with the annual data, that is, consider α-efficiency corresponding to several values $\alpha = 0.1, 0.5, 1, 10, 15$ (only 1991) and 10^3. Therefore, Tables 9 and 10 represent the results of analyses by the basic DEA models and the GDEA model.

As shown in tables, the GDEA model with $\alpha = 0.1$ provides FDH efficiency. It means that there is no change in α-efficient DMUs for smaller α than 0.1. In addition, the GDEA model with $\alpha = 10$ yields BCC efficiency in Table 9, while $\alpha = 15$ does in Table 10. Also, there is no change in α-efficiency of DMUs, even if taking greater α than 10 or 15. Moreover, CCR-efficiency can be conducted by taking α sufficiently large in the GDEA model adding the constraint $x_o^T v = y_o^T \mu$. From this fact, we see that the number of efficient DMUs decreases as a parameter α increases in general. Particularly, note the α-efficiency for $\alpha = 0.5$ and 1: This represents an intermediate efficiency between FDH-efficiency and BCC-efficiency.

In practice, among decision making problems, there exist the cases that it is impossible to correspond to a special value judgments of decision makers such as the CCR efficiency, the BCC efficiency. In contrast to

Table 8
Input and output values for 13 Mexican banks, 1990–1991 (billions of nominal pesos)

Bank	1990			1991		
	Deposits	Non-interest expense	Interest income plus non-interest income	Deposits	Non-interest expense	Interest income plus non-interest income
(1) Banamex	35313.90	2500.88	14247.10	57510.90	3670.33	15764.60
(2) Bancomer	34504.60	2994.70	12682.10	59965.00	3872.40	15877.00
(3) Serfin	30558.20	1746.50	11766.40	46987.20	2709.20	12694.10
(4) Intermac	7603.53	1011.40	3422.40	13458.00	1165.20	4212.20
(5) Cremi	1977.18	1628.80	2889.10	5108.97	760.60	2102.70
(6) Bancreser	2405.00	140.70	1050.50	3314.32	190.80	1681.10
(7) MercNort	2146.06	338.30	1320.10	3714.72	463.30	1377.40
(8) BCH	2944.00	260.8	1410.00	3728.00	402.90	1794.10
(9) Confia	1962.34	266.60	1568.00	3324.43	364.90	1944.40
(10) Bancen	1815.73	196.70	946.20	2544.96	242.70	848.80
(11) Promex	1908.23	251.30	1162.80	3080.00	320.40	1251.40
(12) Banoro	1372.78	169.60	598.20	2799.00	224.40	810.50
(13) Banorie	488.17	71.90	340.80	680.88	86.80	373.00

Table 9
DEA Mexican bank analysis, 13 banks, 1990

Bank	1990										
	CCR		BCC		FDH		GDEA				
	θ	Class	θ	RTS	θ	Class	$\alpha = 10^3$ ($\boldsymbol{x}_o^{\mathrm{T}} v = \boldsymbol{y}_o^{\mathrm{T}} \mu$)	$\alpha = 10$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.1$
(1) Banamex	0.816	NE	1.000	D	1.000	E	−123.46	0.00	0.00	0.00	0.00
(2) Bancomer	0.646	NE	0.890	–	1.000	E	−744.67	−7282.88	−358.41	0.00	0.00
(3) Serfin	0.902	NE	1.000	D	1.000	E	−11.88	0.00	0.00	0.00	0.00
(4) Intermac	0.573	NE	0.809	–	1.000	E	−285.50	−1648.99	0.00	0.00	0.00
(5) Cremi	1.000	E	1.000	C	1.000	E	0.00	0.00	0.00	0.00	0.00
(6) Bancreser	1.000	E	1.000	C	1.000	E	0.00	0.00	0.00	0.00	0.00
(7) MercNort	0.750	NE	0.757	–	0.914	NE	−126.73	−1078.91	−149.92	−102.55	−19.69
(8) BCH	0.829	NE	0.837	–	1.000	E	−70.89	−390.60	−11.27	−0.08	0.00
(9) Confia	1.000	E	1.000	C	1.000	E	0.00	0.00	0.00	0.00	0.00
(10) Bancen	0.778	NE	0.803	–	1.000	E	−94.29	−390.09	−8.06	0.00	0.00
(11) Promex	0.782	NE	0.797	–	1.000	E	−79.50	−506.79	−29.08	−6.76	0.00
(12) Banoro	0.588	NE	0.644	–	1.000	E	−299.20	−606.52	−12.81	0.00	0.00
(13) Banorie	0.862	NE	1.000	I	1.000	E	−58.55	0.00	0.00	0.00	0.00

Output is total interest and non-interest income; inputs are total deposits and non-interest expense. E: efficient; D: decreasing returns to scale (RTS); I: increasing returns to scale; NE: not efficient; C: constant returns to scale.

Table 10
DEA Mexican bank analysis, 13 banks, 1991

Bank	1991											
	CCR		BCC		FDH		GDEA					
	θ	Class	θ	RTS	θ	Class	$\alpha = 10^3$ ($\boldsymbol{x}_o^{\mathrm{T}} v = \boldsymbol{y}_o^{\mathrm{T}} \mu$)	$\alpha = 15$	$\alpha = 10$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.1$
(1) Banamex	0.531	NE	1.000	D	1.000	E	−181.32	0.00	0.00	0.00	0.00	0.00
(2) Bancomer	0.511	NE	1.000	D	1.000	E	−281.95	0.00	0.00	0.00	0.00	0.00
(3) Serfin	0.532	NE	1.000	D	1.000	E	−136.52	0.00	0.00	0.00	0.00	0.00
(4) Intermac	0.569	NE	0.908	–	1.000	E	−257.11	−717.26	0.00	0.00	0.00	0.00
(5) Cremi	0.704	NE	0.772	–	1.000	E	−282.58	−3134.25	−1957.76	0.00	0.00	0.00
(6) Bancreser	1.000	E	1.000	C	1.000	E	0.00	0.00	0.00	0.00	0.00	0.00
(7) MercNort	0.634	NE	0.638	–	0.892	NE	−284.80	−4371.50	−2999.54	−385.14	−212.60	−42.31
(8) BCH	0.826	NE	0.828	–	0.906	NE	−112.88	−1481.79	−982.50	−99.34	−60.03	−15.61
(9) Confia	1.000	E	1.000	C	1.000	E	0.00	0.00	0.00	0.00	0.00	0.00
(10) Bancen	0.592	NE	0.612	–	1.000	E	−253.70	−1621.77	−1075.07	−50.54	0.00	0.00
(11) Promex	0.705	NE	0.715	–	1.000	E	−191.64	−2262.34	−1504.08	−74.49	0.00	0.00
(12) Banoro	0.535	NE	0.554	–	1.000	E	−295.19	−1410.08	−934.00	−80.67	−5.37	0.00
(13) Banorie	0.937	NE	1.000	I	1.000	E	−73.42	0.00	0.00	0.00	0.00	0.00

Output is total interest and non-interest income; inputs are total deposits and non-interest expense. E: efficient; D: decreasing returns to scale (RTS); I: increasing returns to scale; NE: not efficient; C: constant returns to scale.

the existing DEA models, the GDEA model can incorporate his/her various value judgment by changing a parameter α, and then several kinds of efficiency of the basic DEA models can be measured in a unified way on the basis of the GDEA model. Furthermore, the relationships among efficiency for these models become transparent by considering GDEA.

6. Conclusions

In this paper, we suggested the GDEA model based on parametric domination structure, and defined α-efficiency in the GDEA model. In addition, we investigated theoretical properties on relationships between the GDEA model and existing DEA models, specifically, the CCR model, the BCC model and the FDH model. And then, it was proved that the GDEA model makes it possible to evaluate efficiencies of several DEA models in a unified way, and to incorporate various preference structures of decision makers. Through a numerical example, it has been shown that the mutual relations among all DMUs can be grasped by varying α in the GDEA model. Furthermore, we proposed the GDEA_D model based on production possibility as a dual approach to GDEA, and defined α_D-efficiency in the GDEA_D model. Also, we clarified the relations between the GDEA_D model and existing DEA dual models, and interpreted the meaning of an optimal value to the problem (GDEA_D). As a result, it is possible to make a quantitative analysis for inefficiency on the basis of surplus of inputs and slack of outputs. Moreover, through an illustrative example, it has been shown that GDEA_D can reveal domination relations among all DMUs. It is expected from the obtained results in this study that GDEA is useful for evaluating the efficiency of complex management systems in business, industry and social problems.

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