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Crashing PERT networks using mathematical programming

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Abstract

The paper introduces and develops a method for investigating the application of mathematical programming to the concept of crashing in Program Evaluation and Review Technique (PERT). The main objective is the minimization of the pessimistic time estimate in PERT networks by investing additional amounts of money in the activities on the critical path. The constructed mathematical model, which is built in terms of additional amounts of money that must be invested, shows that minimizing the pessimistic time decreases project duration and, at the same time, reduces its variance. The result of applying the model showed that the probability of realizing the terminal node is increased.

Keywords: Project management; PERT; Crashing; Networks; Non-linear programming

1. Introduction

Network crashing was originally developed along with the critical path method (CPM) for planning and controlling large scale projects. The objective of network crashing in CPM is to find which activities should be crashed with the use of additional resources if the duration of the project must be shortened. Crashing in CPM means selecting the lowest cost slope activity or activities, which will shorten the critical path(s). This procedure is repeated until the project has been shortened sufficiently or the cost to shorten the project exceeds the benefits to be derived. The idea of crashing a project was not developed in PERT where stochastic time estimates were used. Probability estimates for completing a project at a specific date were the closest that PERT came to addressing the problem.

Crashing PERT networks was investigated by a number of researchers. George and Schou [1] have explored effective rules for expediting the PERT networks in order to find the activity that must be crashed first.

Samman tried to solve this problem by heuristic methods [2]. Keefer and Verdini tried to enhance the time estimation of PERT activity parameters. This enhancement gave more accuracy than the PERT formulas currently being used [3]. Although several researchers have criticized the time distribution of PERT networks [4,5], it is still considered an effective tool for scheduling probabilistic projects. Cho and Yum [6] developed a new method for evaluating what they called the "Uncertainty Importance Measure". It is the researchers' intention to explore some of the problems in developing a rationale for crashing stochastic networks. In more specific words, the aim of this study is to develop an approach for crashing the pessimistic time in PERT networks by investing additional amounts of money in order to reduce project duration and its variance. This will increase the probability of completing the project in a given time.

The concept of crashing in CPM is applied to PERT networks. In order to reduce the duration of a project that has to be completed within a specified time, additional amounts of money must be invested. Time reduction is necessary in order to avoid tardiness.

2. Model construction

According to the traditional PERT technique the probability of a certain project meeting a specific schedule time T_s, for a particular event is equal to $\varnothing(Z)$, where;

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$$Z = \frac{T_s - \mu_{TE}}{\sigma_{TE}} \tag{1}$$

where

$$\mu_{TE} = \sum_{i=1}^{n} t_{e_i} = t_{e1} + t_{e2} + \cdots + t_{en} \tag{2}$$

$$\sigma_{TE} = \sqrt{\sum_{i=1}^{n} \sigma^2_{t_{e_i}}} = \sqrt{\sigma^2_{t_{e1}} + \sigma^2_{t_{e2}} + \cdots + \sigma^2_{t_{en}}} \tag{3}$$

$$t_{e_i} = \frac{a_i + 4m_i + b_i}{6} \tag{4}$$

$$\sigma^2_{t_{e_i}} = \frac{(b_i - a_i)^2}{36} \tag{5}$$

The amount of money is r_i, where $i = 1, 2, n$ and n is the total number of activities which lie on the critical path. This amount of money will be invested in each activity, in critical path activities possible, to reduce the pessimistic time from b to $\hat{b}$, the expected project duration from μ_{TE} to $\hat{\mu}_{TE}$ and the standard deviation from σ_{TE}) to $\hat{\sigma}_{TE}$ respectively.

Assume that for a certain project, the critical path contains n activities, and the amount of money to be invested in that critical path is M units of money. After investing additional amounts of money, the new expected duration and the new variance for each activity would be

$$\hat{t}_{e_i} = \frac{a_i + 4m_i + \hat{b}_i}{6} \tag{6}$$

$$\hat{\sigma}^2_{t_{e_i}} = \frac{\left(\hat{b}_i - a_i\right)^2}{36} \tag{7}$$

The new expected duration for all activities that lie on the critical path will be as follows

$$\begin{aligned} \hat{t}_{e1} &= \frac{a_1 + 4m_1 + \hat{b}_1}{6} \\ \hat{t}_{e2} &= \frac{a_2 + 4m_2 + \hat{b}_2}{6} \\ &\vdots \\ \hat{t}_{en} &= \frac{a_n + 4m_n + \hat{b}_n}{6} \end{aligned} \tag{8}$$

The new variance for all activities that lie on the critical path will be as follows

$$\begin{aligned} \hat{\sigma}^2_{t_{e1}} &= \frac{\left(\hat{b}_1 - a_1\right)^2}{36} \\ \hat{\sigma}^2_{t_{e2}} &= \frac{\left(\hat{b}_2 - a_2\right)^2}{36} \\ &\vdots \\ \hat{\sigma}^2_{t_{en}} &= \frac{\left(\hat{b}_n - a_n\right)^2}{36} \end{aligned} \tag{9}$$

and the amount of money invested on all the activities that lie on the critical path will be

$$r_1 + r_2 + r_3 + \cdots + r_n = M \tag{10}$$

After investing the amount of money in the activities which lie on the critical path, the probability of realizing the terminal node is equal to $\varnothing(Z)$, where,

$$Z = \frac{T_s - \hat{\mu}_{TE}}{\hat{\sigma}_{TE}} \tag{11}$$

$$\hat{\mu}_{TE} = \sum_{i=1}^{n} \hat{t}_{e_i} \tag{12}$$

and

$$\hat{\sigma}_{TE} = \sqrt{\sum_{i=1}^{n} \hat{\sigma}^2_{t_{e_i}}} \tag{13}$$

Then the probability of realizing the terminal node will be

$$Z = \frac{T_s - \sum_{i=1}^{n}\left(\frac{a_1 + 4m_1 + \hat{b}_1}{6} + \frac{a_2 + 4m_2 + \hat{b}_2}{6} + \cdots + \frac{a_n + 4m_n + \hat{b}_n}{6}\right)}{\sqrt{\sum_{i=1}^{n}\left(\frac{\left(\hat{b}_1 - a_1\right)^2}{36} + \frac{\left(\hat{b}_2 - a_2\right)^2}{36} + \cdots + \frac{\left(\hat{b}_n - a_n\right)^2}{36}\right)}} \tag{14}$$

Since the amount of money invested will reduce the expected time t_e, this reduction will depend on the amount of money invested in each activity. This means the decrease in expected time from t_e to $\hat{t}_e$ will be a function $\varnothing$ of additional investment. The new expected duration for a certain activity will be

$$\hat{t}_{e_i} = t_{e_i} - \varnothing(r_i) \tag{15}$$

The equation of new expected duration becomes:

$$\hat{t}_e = t_e + q_i r_i \quad -\frac{t_{e_i}}{r_i} < q_i < 0 \tag{16}$$

where $q_i < 0$ is the marginal decrease in the level (duration) of activity per unit increase in investment in the activity. In the same way the additional amount r_i will reduce the variance and this reduction depends on the amount of money invested. This means that the decrease is a function ψ of r_i. Then the new level of variance is equal to

$$\hat{\sigma}_i^2 = \sigma_i^2 - \psi(r_i) \tag{17}$$

and the new variance becomes

$$\hat{\sigma}_i^2 = \sigma_i^2 + s_i r_i \quad -\frac{\sigma_i^2}{r_i} < s_i < 0 \tag{18}$$

where $s_i < 0$ is the marginal decrease in the level of the variance per unit increase in the investment.

Reducing the variance by a certain percent is equivalent to reducing the standard deviation by the same percent. Hence, Eq. (18) can be written as follows

$$\hat{\sigma}_i = \sigma_i + s_i r_i \quad -\frac{\sigma_i}{r_i} < s_i < 0 \tag{19}$$

but

$$\sigma_i = \frac{b-a}{6} \quad \text{and} \quad \hat{\sigma}_i = \frac{\hat{b}-a}{6}$$

Then

$$\frac{\hat{b}-a}{6} = \frac{b-a}{6} + s_i r_i \tag{20}$$

Also, from Eq. (16) we get

$$\hat{b} = b + 6 q_i r_i \tag{21}$$

By substituting Eqs. (20) and (21) the following is obtained

$$s_i = q_i \tag{22}$$

which means that the two proportional s_i and q_i must be the same because the amount of money that must be invested in a certain activity will reduce pessimistic time and this time is a part of the activity duration and its variance.

Since decrease in the expected duration is a function of r_i, the new expected duration of the crashed activities that lie on the critical path will be

$$\begin{aligned} \hat{t}_{e1} &= t_{e1} - \varnothing_1(r_1) \\ \hat{t}_{e2} &= t_{e2} - \varnothing_2(r_2) \\ &\vdots \\ \hat{t}_{en} &= t_{en} - \varnothing_n(r_n) \end{aligned} \tag{23}$$

After approximating the $\varnothing$ function, the previous equations becomes

$$\begin{aligned} \hat{t}_{e1} &= t_{e1} + q_1 r_1 & -\frac{t_{e1}}{r_1} &< q_1 < 0 \\ \hat{t}_{e2} &= t_{e2} + q_2 r_2 & -\frac{t_{e2}}{r_2} &< q_2 < 0 \\ &\vdots & &\vdots \\ \hat{t}_{en} &= t_{en} + q_n r_n & -\frac{t_{en}}{r_n} &< q_n < 0 \end{aligned} \tag{24}$$

Since the decrease in the variance is a function of r_i, then the new variance of the crashed activities that lie on the critical path becomes

$$\begin{aligned} \hat{\sigma}_{t_{e1}}^2 &= \sigma_{t_{e1}}^2 - \psi_1(r_1) \\ \hat{\sigma}_{t_{e2}}^2 &= \sigma_{t_{e2}}^2 - \psi_2(r_2) \\ &\vdots \\ \hat{\sigma}_{t_{en}}^2 &= \sigma_{t_{en}}^2 - \psi_n(r_n) \end{aligned} \tag{25}$$

In the same way, and after approximating the ψ function, the previous equations can be written

$$\begin{aligned} \hat{\sigma}_{t_{e1}}^2 &= \sigma_{t_{e1}}^2 + s_1 r_1 & -\frac{\sigma_{t_{e1}}^2}{r_1} &< s_1 < 0 \\ \hat{\sigma}_{t_{e2}}^2 &= \sigma_{t_{e2}}^2 + s_2 r_2 & -\frac{\sigma_{t_{e2}}^2}{r_2} &< s_2 < 0 \\ &\vdots & &\vdots \\ \hat{\sigma}_{t_{en}}^2 &= \sigma_{t_{en}}^2 + s_n r_n & -\frac{\sigma_{t_{en}}^2}{r_n} &< s_n < 0 \end{aligned} \tag{26}$$

After investing the amount r_i in activities that lie on the critical path, the new expected duration and new variance are functions of r_i. The probability of realizing the terminal node is $\varnothing(Z)$, where

$$Z = \frac{T_s - \hat{\mu}_{TE}}{\hat{\sigma}_{TE}} \tag{27}$$

where

$$\hat{\mu}_{TE} = \hat{t}_{e1} + \hat{t}_{e2} + \cdots + \hat{t}_{en}$$

and

$$\hat{\sigma}_{TE} = \sqrt{\hat{\sigma}_{t_{e1}}^2 + \hat{\sigma}_{t_{e2}}^2 \cdots + \hat{\sigma}_{t_{en}}^2}$$

Then Z becomes

$$Z = \frac{T_s - \{\hat{t}_{e1} + \hat{t}_{e2} + \cdots + \hat{t}_{en}\}}{\sqrt{\{\hat{\sigma}^2_{t_{e1}} + \hat{\sigma}^2_{t_{e2}} \cdots + \hat{\sigma}^2_{t_{en}}\}}} \tag{28}$$

where

$$\hat{t}_{e_i} = t_{e_i} + q_i r_i$$

and

$$\hat{\sigma}^2_{t_{e_i}} = \sigma^2_{t_{e_i}} + s_i r_i$$

Then,

$$Z = \frac{T_s - \{t_{e1} + q_1 r_1 + t_{e2} + q_2 r_2 + \cdots + t_{en} + q_n r_n\}}{\sqrt{\{\sigma^2_{t_{e1}} + s_1 r_1 + \sigma^2_{t_{e2}} + s_2 r_2 + \cdots + \sigma^2_{t_{en}} + s_n r_n\}}} \tag{29}$$

Since $\sum_{i=1}^n t_{e_i}$ and $\sum_{i=1}^n \sigma^2_{t_{e_i}}$ are the same before and after crashing

$$\sum_{i=1}^n t_{e_i} = t_{e1} + t_{e2} + \cdots + t_{en} = \text{constant} = C_1 \tag{30}$$

$$\sum_{i=1}^n \sigma^2_{t_{e_i}} = \sigma^2_{t_{e1}} + \sigma^2_{t_{e2}} + \cdots + \sigma^2_{t_{en}} = \text{constant} = C_2 \tag{31}$$

Then

$$Z = \frac{T_s - \left\{C_1 + \sum_{i=1}^n q_i r_i\right\}}{\sqrt{C_2 + \sum_{i=1}^n s_i r_i}} \tag{32}$$

where

$$\frac{-t_{e_i}}{r_i} < q_i < 0$$

and

$$\frac{-\sigma_i^2}{r_i} < s_i < 0$$

$$0 \le r_i \le \overline{r_i}$$

where $\overline{r_i}$ is the upper limit for the amount of money to be invested in activity i.

Eq. (32) is the required mathematical model, which represents the objective function and the constraints. The value of q_i and s_i are specified by experts having high knowledge in the nature of project activities. The variable in this model is r_i. The whole model can be rewritten as follows

$$\text{Max}\, Z = \frac{T_s - C_1 - \sum_{i=1}^n q_i r_i}{\sqrt{C_2 + \sum_{i=1}^n s_i r_i}} \tag{33}$$

such that

$$0 \le r_i \le \overline{r_i} \quad \text{and} \quad \sum_{i=1}^n r_i \le M$$

3. Linearizing the mathematical model

In order to investigate the application of mathematical programming to the concept of crashing in PERT networks the mathematical model Eq. (33) will be linearized by using Taylor's first order expansion for multidimensional function [7]. Then, the linearized model will be

$$\begin{aligned}\text{Max } Z = & \left[T_s - C_1 - \sum_{i=1}^n q_i \overline{r_i}\right]\left[C_2 + \sum_{i=1}^n s_i \overline{r_i}\right] - \frac{1}{2} \\ & + \sum_{i=1}^n \Bigg[- q_i[C_2 + s_i\overline{r_i}]^{-\frac{1}{2}} - \frac{1}{2}[C_2 + s_i\overline{r_i}]^{-\frac{3}{2}}[s_i]\big[T_s \\ & - C_1 - q_i\overline{r_i}\big]\Bigg][r_i - \overline{r_i}]\end{aligned}$$

subject to $0 \le r_i \le \overline{r_i}$

$$\sum_{i=1}^n r_i \le M \tag{34}$$

where, $\overline{r_i}$ is the upper limit of money to be invested on activity i, M is the total amount of money to be invested on all activities that lie on the critical path, and n is the number of activities that lie on the critical path. From Eq. (34), the first term is constant and depends on $\overline{r_i}$.

$$\begin{aligned}&\left[T_s - C_1 - \sum_{i=1}^n q_i\overline{r_i}\right]\left[C_2 + \sum_{i=1}^n s_i\overline{r_i}\right]^{\frac{1}{2}} = \text{constant} \\ &= \alpha\end{aligned} \tag{35}$$

Then, Eq. (34) can be written as subject to

$$\begin{aligned}\text{Max } Z = & \alpha + \sum_{i=1}^n \Bigg[- q_i[C_2 + s_i\overline{r_i}]^{-\frac{1}{2}} - \frac{1}{2}[C_2 + s_i\overline{r_i}]^{-\frac{3}{2}} \\ & [s_i][T_s - C_1 - q_i\overline{r_i}]\Bigg][r_i - \overline{r_i}]\end{aligned} \tag{36}$$

Subject to

$$0 \le r_i \le \overline{r_i}$$
$$\sum_{i=1}^{n} r_i \le M$$

Eq. (36) is a linear equation, and can be solved by using any linear programming package. Since α in Eq. (36) is constant, then it can be written as follows subject to

$$\text{Max } Z = \sum_{i=1}^{n} \left[-q_i[C_2 + s_i\overline{r_i}]^{-\frac{1}{2}} - \frac{1}{2}[C_2 + s_i\overline{r_i}]^{-\frac{3}{2}}[s_i] [T_s - C_1 - q_i\overline{r_i}]\right][r_i - \overline{r_i}] \tag{37}$$

Subject to

$$0 \le r_i \le \bar{r}_i$$
$$\sum_{i=1}^{n} r_i \le M$$

Eq. (37) is the linearized constructed model that will be applied for crashing PERT networks. A program was developed to substitute the values of s_i, q_i, T_s, C_1, C_2 and $\bar{r}_i$ in the linearized constructed model. The output of this program is a linear equation, which is the objective function of the linearized constructed model, which will be in the following form

$$Z = \alpha + \beta_1 r_1 + \beta_2 r_2 + \cdots + \beta_n r_n$$

where α is a constant, β_1 the coefficient of the additional amount of money which must be invested on the first activity which lay on the critical path, and n is the number of activities that must be crashed. The constraint of the objective function will be

$$0 \le r_i \le \overline{r_i}$$
$$\sum_{i=1}^{n} r_i \le M$$

where $\overline{r_i}$ is the upper limit of the money to be invested in a certain activity, and it is specified by experts according to the value of M, the total amount of money to be invested in the critical path specified by top management. Maximization can be achieved by using any linear programming package.

4. Algorithm development

In any project there can be one or more critical path(s). However, when using PERT the critical path is the path with the least probability of completion. In this paper a model is developed for the critical path only. The following information must be known before applying the model:

1. Determine the critical path.
2. Specify the activity to be crashed that lies on the critical path — this is done by experts.
3. Specify the values of q_i and s_i. These values are also determined by experts who know the nature of the crashed activity.
4. Specify the value of required schedule time T_s.
5. Specify the upper limit of money to be invested on the crashed activity $\overline{r_i}$. Experts who know the minimum level at which the activity can be crashed determine this limit.
6. Specify the limited availability of the total amount of money to be invested M. Top management of the organization specifies this value.
7. Determine the value of C_1, i.e. the project duration.
8. Determine the value of C_2, i.e. the summation of variance of the activities on the critical path C_1 and C_2.

The algorithm steps of the constructed model are listed below:

1. Determine the equation of the objective function of the linearized constructed model Eq. (37) by substituting the values of C_1, C_2, T_s, q_i and s_i. Also, choose the origin as a starting point to linearize around it.
2. Maximize the objective function in step 1 above.
3. From step 2, find the values of $\overline{r_i}$ (the upper limit of money to be invested on activity i).
4. From step 3, substitute the value of $\overline{r_i}$ in the constructed model to find the value of $Z(r_i)$.
5. Find the value of $\varnothing Z(r_i)$, by using the table of normal standardized distribution.
6. Linearize the model around r_i, which has been determined in step 3, and then repeat steps 1 and 2.
7. If the new value of r_i in step 6 is equal to r_i in step 3, then stop, otherwise find the new value of $Z(r_i)$.
8. Compare the value of $Z(r_i)$ in step 7 with the value of $Z(r_i)$ in step 4. If the difference between them is very small, then stop, otherwise linearize the model around r_i, (which has been determined in step 6) and then repeat step 1 and step 2. Then find the values of r_i.
9. If the value of r_i, in step 8 is the same as in step 6 then stop, otherwise find the new value of $Z(r_i)$.
10. Compare the value of $Z(r_i)$ in step 9 with the value of $Z(r_i)$ in step 7, if the difference between them is very small, then stop, and otherwise repeat the steps of the algorithm until the difference between $Z(r_i)$ becomes negligible.

The above algorithm was applied to a large number of networks. The following is a case study to demonstrate how the model works.

5. The case study

The developed algorithm was evaluated using a large number of networks. In this section we present a case study consisting of 42 activities to demonstrate how the model can be applied to decrease project duration.

Table 1 shows the three time estimates of the activities: optimistic (a), most probable (m) and pessimistic times. While Fig. 1 depicts the case study network using activity on arrow (AoA) notations. Table 2 shows the values of q_i, and s_i, which are specified by experts who have sufficient knowledge about the nature of the activities to be crashed.

By substituting the values of q_i, s_i and $\bar{r}_i$ in the developed model, we get the equation of the objective function.

Table 1
The three time estimates for each activity of the case study

Activity No.	a, m, b	Activity No.	a, m, b	Activity No.	a, m, b
1	5, 12, 35	15	2, 3, 5	29	26, 29, 31
2	2, 5, 6	16	12, 14, 15	30	30, 32, 35
3	5, 8, 9	17	12, 14, 18	31	12, 14, 19
4	10, 12, 15	18	12, 15, 18	32	23, 29, 39
5	2, 5, 9	19	12, 15, 19	33	25, 29, 38
6	2, 5, 8	20	15, 16, 19	34	12, 14, 15
7	5, 9, 14	21	15, 18, 19	35	22, 28, 35
8	5, 8, 9	22	9, 10, 12	36	23, 24, 30
9	8, 9, 16	23	12, 15, 17	37	10, 11, 15
10	5, 6, 8	24	12, 13, 19	38	23, 23, 24
11	12, 13, 15	25	21, 22, 29	39	30, 32, 36
12	5, 9, 16	26	15, 17, 19	40	30, 35, 36
13	5, 8, 11	27	11, 16, 18	41	30, 30, 32
14	13, 15, 18	28	15, 19, 21	42	25, 26, 29

Next, by specifying the values of $\overline{r_i}$ and M, we get the constraint of the objective function. The whole model can be written as

$$
\begin{array}{lll}
\multicolumn{3}{l}{\mathrm{Max}\,Z =} \\
0.00001285r_1 & +0.00003855r_4 & +0.00001413r_9 \\
0.00002956r_{14} & +0.00005269r_{19} & +0.0000424r_{21} \\
0.0000642r_{20} & +0.00004626r_{25} & +0.00002313r_{29} \\
0.00005783r_{30} & +0.00004626r_{32} & +0.00007068r_{33} \\
0.0000848r_{35} & +0.00008996r_{36} & +0.0000257r_{39} \\
0.00003213r_{40} & +0.00003855r_{41} & +0.00000514r_{42}
\end{array}
$$

subject to

$$
\begin{array}{lll}
0 \le r_1 \le 5000 & 0 \le r_{20} \le 1200 & 0 \le r_{35} \le 1500 \\
0 \le r_4 \le 1000 & 0 \le r_{25} \le 1800 & 0 \le r_{36} \le 1100 \\
0 \le r_9 \le 3000 & 0 \le r_{29} \le 800 & 0 \le r_{39} \le 2200 \\
0 \le r_{14} \le 2000 & 0 \le r_{30} \le 750 & 0 \le r_{40} \le 500 \\
0 \le r_{19} \le 1000 & 0 \le r_{32} \le 3200 & 0 \le r_{41} \le 850 \\
0 \le r_{21} \le 200 & 0 \le r_{33} \le 2000 & 0 \le r_{42} \le 6000
\end{array}
$$

$$
\begin{aligned}
& r_1 + r_4 + r_9 + r_{14} + r_{19} + r_{21} + r_{20} + r_{25} + r_{29} + r_{30} \\
& \quad + r_{32} + r_{33} + r_{35} + r_{36} + r_{39} + r_{40} + r_{41} + r_{42} \le 30000
\end{aligned}
$$

The solution is

$$
\begin{array}{lll}
R_1 = 5000 & r_{20} = 1200 & r_{35} = 1500 \\
R_4 = 1000 & r_{25} = 1800 & r_{36} = 1100 \\
R_9 = 3000 & r_{29} = 800 & r_{39} = 2200 \\
R_{14} = 2000 & r_{30} = 750 & r_{40} = 500 \\
R_{19} = 1000 & r_{32} = 3200 & r_{41} = 850 \\
R_{21} = 2000 & r_{33} = 2000 & r_{42} = 100
\end{array}
$$

Then, we continue the steps of the algorithm, by finding the value of the objective function Z and substituting in

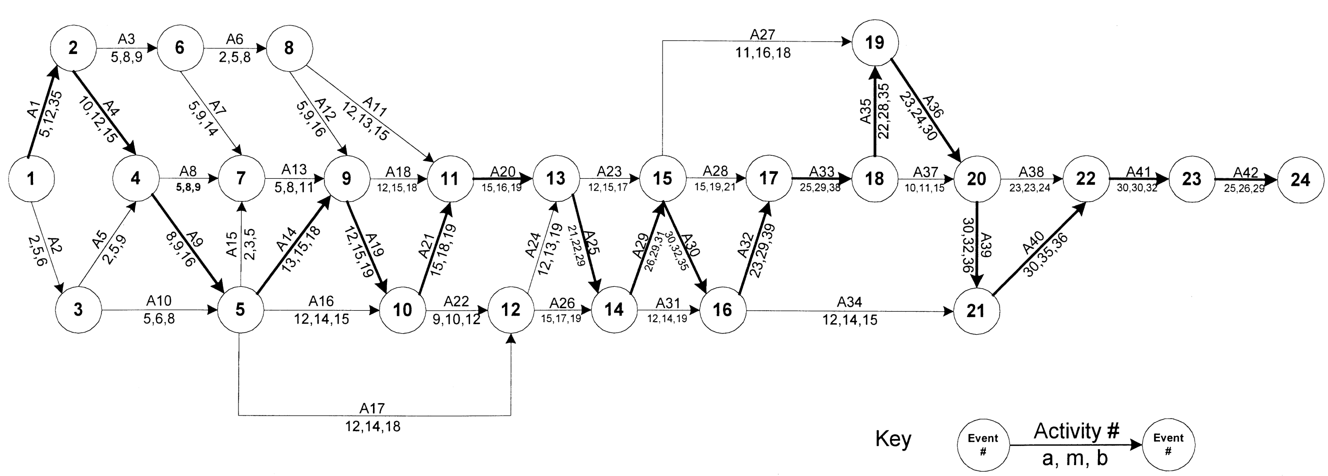

Fig. 1. Activity on arrow (AoA) network for the case study.

Eq. (37), we get $Z = -0.36476$ and $\varnothing(Z) = 35\%$. Also, we continue the steps of the algorithm by linearizing the model around $\overline{r_i}$, (which were determined above) to get the following objective function

$$\begin{aligned}\text{Max } Z = \\ &0.00001296r_1 &&+0.00003876r_4 &&+0.00001422r_9\\ &0.0000298r_{14} &&+0.00005308r_{19} &&+0.00006437r_{21}\\ &0.0000427r_{20} &&+0.0000468r_{25} &&+0.00002319r_{29}\\ &0.00005818r_{30} &&+0.0000472r_{32} &&+0.00007211r_{33}\\ &0.0000863r_{35} &&+0.0000912r_{36} &&+0.0000259r_{39}\\ &0.0000322r_{40} &&+0.0000387r_{41} &&+0.000005141r_{42}\end{aligned}$$

subject to the same constraints above.

The solution is the same when we linearized around

$$\begin{bmatrix}0\\0\\.\\0\end{bmatrix}.$$

Then the optimal solution will be the same solution when we linearize around the origin. Table 3, summarizes the results for the pessimistic time, project duration and variance for the activities of the critical path.

6. Discussion

In this paper the concept of crashing in CPM has been applied to PERT networks by investing additional amounts of money. A mathematical model was constructed to achieve this purpose. The objective of the constructed model was to increase the probability of realizing the last node by minimizing pessimistic time, which led to a decrease in project duration and standard deviation.

The model was tested on several networks, and the results were as expected. By investing additional amounts of money the pessimistic time and, the probability of completing the whole project increased from 20.7% to 35%. Moreover, the schedule time T_s decreased from 421 days to 413 days.

The increase in probability is due to increasing the area to the left of the schedule time T_s. The probability of completing the project increases as the total investment increases, because project duration and variance

Table 2
Values of q_i and s_i for the activities to be crashed

Activity No.	q_i	s_i	Activity No.	q_i	s_i
1	−0.0001	−0.0001	30	−0.00045	−0.00045
4	−0.0003	−0.0003	32	−0.00036	−0.00036
9	−0.00011	−0.00011	33	−0.00055	−0.00055
14	−0.00023	−0.00023	35	−0.00066	−0.00066
19	−0.00041	−0.00041	36	−0.0007	−0.0007
21	−0.00005	−0.00005	39	−0.0002	−0.0002
20	−0.00033	−0.00033	40	−0.00025	−0.00025
25	−0.00036	−0.00036	41	−0.0003	−0.0003
29	−0.00018	−0.00018	42	−0.00004	−0.00004

Table 3
Model results

Activity	a	m	B	σ	t_e	μ_{TE}	M	$\hat{b}$	$\hat{\sigma}$	$\hat{t}_e$	$\hat{\mu}_{TE}$
						421	30,000				413
1	5	12	35	5	14.66			32	4.5	14.16	
4	10	12	15	0.83	12.16			13.2	0.53	11.86	
9	8	9	16	1.33	10.00			14	1	9.66	
14	13	15	18	0.83	15.16			15.24	0.373	14.70	
19	12	15	19	1.16	15.16			16.54	0.75	14.75	
21	15	18	19	0.66	17.66			18.40	0.56	17.56	
20	15	16	19	0.66	16.33			16.62	0.27	15.93	
25	21	22	29	1.33	23.00			25.11	0.68	22.30	
29	26	29	31	0.83	28.80			30.13	0.68	28.60	
30	30	32	35	0.83	32.16			32.97	0.49	31.80	
32	23	29	39	2.66	29.66			32.08	1.51	28.51	
33	25	29	38	2.16	29.83			31.4	1.06	28.70	
35	22	28	35	2.16	28.16			29.06	1.17	27.17	
36	23	24	30	1.16	24.80			25.38	0.39	24.06	
39	30	32	36	1	32.33			33.36	0.56	31.89	
40	30	35	36	1	34.33			35.25	0.875	34.2	
41	30	30	32	0.33	30.33			30.47	0.078	30.78	
42	25	26	29	0.66	26.33			27.56	0.42	26.09	

will decrease at the same time up to the crash limits. Crashing PERT is necessary to reduce time in order to avoid tardiness.

7. Conclusions

Activities in PERT networks have three time estimates; optimistic, most probable and pessimistic. A mathematical model for crashing PERT networks based on the pessimistic time estimate using mathematical models has been constructed and linearized. In this paper the pessimistic time estimate was minimized based on investment of additional funds for activities which are on the critical path.

The results of applying the proposed mathematical model to a case study showed that minimizing pessimistic time led to a decrease in project duration and, at the same time, reduced variance. This in turn resulted in increased probability of accomplishing the project. Unfortunately, the results of the developed model could not be compared with other results since none of the researchers have worked in this area; it is left for future researchers to compare their results with ours. The constructed model can be used to avoid tardiness. This approach can help project planners investigate the trade-off between time and cost in PERT networks.

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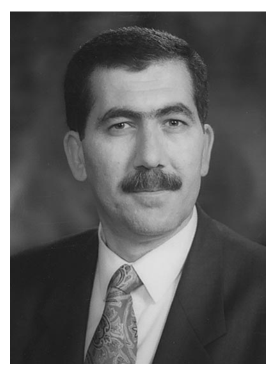

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